Polynomial Adjustment Costs in FRB/US

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The adjustment dynamics of most major nonfinancial variables in FRB/US are based on a framework of polynomial adjustment costs, or PAC. This is a model, developed by Tinsley (1993), in which decisions are driven by expectations but constrained by adjustment costs. This note describes what PAC is, how it is derived and how it is estimated and implemented within FRB/US.

Description

The PAC model can be presented in different ways, depending on the purpose. A form that is simple to interpret is the "decision rule":

$$\Delta y_t = a_0 \left(y^*_{t-1} - y_{t-1} \right) + \sum_{k=1}^{m-1} a_k \Delta y_{t-k} + E_{t-1} \sum_{j=0}^{\infty} d_j \Delta y^*_{t+j} \qquad \textbf{(1)}$$

where y is the dependent variable, y^* represents its desired, target or equilibrium value (we use the terms interchangeably), Δ is the first difference operator and E_{t-1} represents expectations based on information available at t-1. Thus the equation decomposes the determinants of Δy into three elements: the lagged gap between the level of y and its equilibrium value, lagged values of Δy, and expected future values of Δy^*. The d_j coefficients on leads of Δy^* are transformations of the a_k ($k = 0, ..., m-1$) coefficients on lags of y, as discussed below. In FRB/US notation, the level of the target, y^*, is represented by a variable in which the first letter is Q. The expected sum of future values of Δy^* is represented by a variable in which the first letter is Z.

This approach resembles FRB/US's precursor, MPS, and similar models in that it

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combines short-term dynamics with long-term error correction mechanisms. The close fit to the data associated with this approach is maintained. The main difference from simple error-correction models is the inclusion of terms reflecting expected growth in the target. In traditional error-correction models, lagged explanatory variables often represent a mixture of expectations and previous shocks to which agents are gradually adjusting. This ambiguity makes it difficult to address important questions, such as the effect of changes in expectations. In contrast, the PAC framework provides a means of separately identifying the effects of expectations and adjustment costs.

Derivation

Several observationally-equivalent cost functions can be used to derive the PAC specification. One such cost function,

$$C_t = \sum_{i=0}^{\infty} \beta^i \left[(y_{t+i} - y^*_{t+i})^2 + \sum_{k=1}^{m} b_k \left((1-L)^k y_{t+i}\right)^2 \right] \qquad \textbf{(2)}$$

penalizes both deviations of a variable *y* from its desired value y^* and changes in *m* time derivatives of the variable *y*. In this and following equations, future-dated variables should be interpreted as expected values even when the expectations operator has been suppressed. β is a discount factor on future penalties, assumed to equal 0.98, and b_k, k=1, ... , m, are cost parameters.

Many other researchers have worked with cost equations similar to (2), but restricting *m* to equal 1. For examples and discussion see Kennan (1979), Rotemberg (1982) and Nickell (1985). The extra terms permitted by PAC imply inclusion of lagged changes in the dependent variable: the $\sum_{k=1}^{m-1} a_k \Delta y_{t-k}$ expression in **(1)**. Their inclusion provides a better fit to the data, particularly in regard to short-term dynamics. Many equations in FRB/US have values of *m* between 2 and 4.

The Appendix shows how the decision rule **(1)** is derived from the cost function **(2)**. In brief, minimization of costs yields the first-order condition,

$$(y_t - y_t^*) + \sum_{k=1}^{m} b_k \, [\, (1-L) \, (1-\beta F) \,]^k \, y_t = 0 \tag{3}$$

where L and F are the lag and lead operators, respectively; $(L = F^{-1})$. Equation (3) can be written more compactly as

$$A(\beta F) \; A(L) \; y_t = c \; y_t^*, \tag{4}$$

where A is a polynomial in the lag and lead operators of order m; that is, $A(L) = 1 + \alpha_1 L + \ldots + \alpha_m L^m$ and $A(\beta F) = 1 + \alpha_1 \beta F + \ldots + \alpha_m \beta^m F^m$; and c is a constant. After some algebra, this yields **(1)**. The coefficients in the decision rule **(1)** are transformations of the α parameters in A (which in turn are transformations of β and the b parameters in the cost equation **(2)**). Specifically,

$$a_0 = d_0 = A(1) = 1 + \sum_{j=1}^{m} \alpha_j;$$

for $k = 1, 2, \ldots, m-1$

$$a_k = -\sum_{j=k+1}^{m} \alpha_j$$

and, for $j = 1, 2, \ldots \infty$

$$d_j = 1 - A(1) A(\beta) \sum_{i=0}^{j-1} \xi' G^i \xi$$

where the matrix G is also a function of the discount factor β and the α adjustment coefficients. Its structure is provided in equation **(A. 38)** of the Appendix. Pre and post multiplication by the selection vector ξ selects the top left element of G^i.

Estimation

The first step in the estimation of a PAC equation is construction of a model determining the target, y^*. This typically consists of a stationary component, y^{0*} and a trending component, y^{1*}. That is,

$$y^* = \gamma y^{0*} + y^{1*} \tag{5}$$

The coefficient γ on the stationary component is estimated within the PAC equation, as discussed below. In contrast, the trending component is specified before the PAC equation is estimated as a function of one or more variables, with coefficients that are constrained in accord with theory or are estimated from cointegrating equations.

In the second step, forecasting models for the components of y^* are estimated. Specifically, suppose that y^* is an element of the information vector *z*, which also includes other variables useful for forecasting y^*. A vector autoregression (VAR) can then be used to predict future levels of z as a linear combination of past levels. That is,

$$z_{t+1} = H z_t \tag{6}$$

Forecasts into the indefinite future are obtained by repeated application of this equation:

$$z_{t+i} = H^i z_t \tag{7}$$

Decompose z into a vector z^0 used for forecasting the stationary component of the target y^{0*} and a vector z^1 used for forecasting the trending component of the target y^{1*}. Substituting the VAR forecasts for expected changes in the target in equation **(1)** and converting the infinite leads into a finite form gives an equation that is a function of observable variables:

$$\Delta y_t = a_0 \left(y^{1*}_{t-1} - y_{t-1} \right) + \sum_{k=1}^{m-1} a_k \Delta y_{t-k} + \gamma h_0^{\prime} z^0_{t-1} + h_1^{\prime} z^1_{t-1} \tag{8}$$

the coefficient γ represents the contribution of stationary elements of the target. The vectors h_0 and h_1 are constructed from the VAR coefficients *H*, the discount factor β and the α adjustment coefficients, as outlined in equations **(A. 74)** and **(A. 82)** of the Appendix. h_0 and h_1 differ because $\gamma h_0^{/} z_{t-1}^{0}$ reflects contributions from both the lagged level and expected values of the stationary component of the target whereas $h_1^{/} z_{t-1}^{1}$ only reflects contributions from expected movements -- the lagged level of the trending component being separately identified.

PAC imposes numerous restrictions on equation **(8)**. These reflect assumptions that lead coefficients are reparameterizations of the lag coefficients (in turn reflecting the symmetry with which the past and the discounted future enter (4)) and that the elements of z^0 and z^1 affect *y* symmetrically, through their effect on y^* . FRB/US equations often impose additional restrictions reflecting assumptions about homogeneity and cointegration.

A simple method of estimating the restricted equation is through an iterative OLS procedure. Given values of the VAR coefficients *H*, the discount factor β, and starting values for a_k (*k = 0* to *m-1*), initial estimates of h_0 and h_1 can be constructed ($\hat{h}_0$ and $\hat{h}_1$). Using these, we can estimate the following linear regression

$$\Delta y_t - \hat{h}_1^{/} z_{t-1}^{1} = a_0 \left(y_{t-1}^{1*} - y_{t-1} \right) + \sum_{k=1}^{m-1} a_k \Delta y_{t-k} + \gamma \, (\hat{h}_0^{/} z_{t-1}^{0}) \qquad \textbf{(9)}$$

This provides an estimate of γ and revised estimates of the a_k coefficients. $\hat{h}_0$ and $\hat{h}_1$ can then be recalculated and another iteration performed. Typically, parameter estimates converge in a few iterations.

The restrictions imposed by PAC can be jointly tested by comparing the residuals from **(9)** with those from an unrestricted regression. The unrestricted regression resembles equation **(8)** without restrictions on the *h* vectors. The order of adjustment costs *m* is determined empirically by testing to see how many lags of the dependent variable are significant, and then including all lags up through the last significant one.

In practice, not all the series modeled in FRB/US exactly fit within the PAC framework, involving some modifications to the specification above. For example, the presence of agents who do not optimize in the forward-looking manner assumed by PAC (perhaps because of liquidity constraints or bounded rationality) is reflected in the inclusion of current income in the consumption equation and cash-flow in the investment equation. Temporary responses to variables outside the target, (for example payroll taxes and the minimum wage in the wage equation) and different speeds of adjustment to different elements of the target (for example, imports and energy costs in the price equation) is reflected in their separate inclusion as regressors.

Model-consistent expectations

Given a set of VAR-based expectations, the model can be estimated as described above and then used for forecasting and policy simulations. The simulations will normally yield different values for the target variables than implied by the VARs. In other words, the information individuals use in forming their expectations is assumed to be limited to the information vector *z* and does not encompass predictions of the model.

For some applications, such as permanent changes in policy rules, VAR expectations have the undesirable feature that persistent expectational errors are made. To avoid this, policy simulations in FRB/US often assume model-consistent expectations.

To simulate the model with model-consistent expectations, equation **(9)** is initially estimated using VAR-based expectations, as discussed above. This generates an initial set of coefficients, VAR-based expectations of y^* and forecasts of the other endogenous variables of the model. As explained below, new expectations of y^* can then be computed. Substituting these for the previous set of expectations, while retaining the a_k, β and γ coefficients from the initial estimation (which remain constant through successive iterations), new forecasts and new expectations can then be computed. Iterations continue until expectations coincide with forecasts.

Specifically, let Z_t represent the contribution from expected movements in the target. Z_t equals the term $Z_t = E_{t-1} \sum_{j=0}^{\infty} d_j \Delta y^*_{t+j}$ in the decision rule **(1)** and the term $\gamma h_0^{/} z^0_{t-1} + h_1^{/} z^1_{t-1}$ in the estimation equations **(8)** and **(9)** under VAR-expectations. To represent Z_t as an expression containing a finite number of leads, begin by assuming *m* terminal conditions for Z_{t+T} ... Z_{t+T+m} where *T* is large. (These could be arbitrary, though it is more efficient to assume a balanced growth path consistent with the exogenous variables of the model.) Coupled with a projected path of y^* from the present to the terminal state, intermediate values of Z can then be calculated recursively from the terminal state back to period *t*. The formula for calculating this, as explained at equation **(A. 90)** of the Appendix, is:

$$Z_t = \sum_{i=1}^{m} \alpha_i \beta^i Z_{t+i} + A(1) \left[\Delta y^*_t - \sum_{k=1}^{m-1} \left(\sum_{j=k}^{m-1} \alpha_{j+1} \beta^{j+1} \right) \Delta y^*_{t+k} \right] \qquad \textbf{(10)}$$

An initial estimate of Z_t will then imply new forecasts of the model's endogenous variables. From these, a new path of Δy^*_{t+k} can be forecast, from which a new estimate of Z_t can be obtained. This process iterates until the expected growth of the target coincides with the forecast.

Having obtained one set of model-consistent projections, the exogenous variables of the model, such as policy parameters, can then be changed. Repetition of the above procedure leads to a new set of model-consistent projections. These can be compared with the initial set to assess the consequences of the policy change.

Algebraic Appendix

Tinsley (1993) concisely presents the algebra of PAC. This appendix provides a more detailed presentation.

Derivation of the decision rule

Consider the cost function **(2)**:

$$C_t = \sum_{i=0}^{\infty} \beta^i \left[\left(y_{t+i} - y^*_{t+i}\right)^2 + \sum_{k=1}^{m} b_k \left((1-L)^k y_{t+i} \right)^2 \right] \qquad \textbf{(A. 11)}$$

Differentiating with respect to y_t yields a first order condition:

$$0 = 2(y_t - y^*_t) + \sum_{i=0}^{\infty} \beta^i \left[b_1 \frac{\partial\left((1-L)y_{t+i}\right)^2}{\partial y_t} + b_2 \frac{\partial\left((1-L)^2 y_{t+i}\right)^2}{\partial y_t} + \ldots + b_m \frac{\partial\left((1-L)^m y_{t+i}\right)^2}{\partial y_t} \right]. \qquad \textbf{(A. 12)}$$

Take the derivatives of the second part of equation **(A. 12)** piece by piece. For k=1,

$$\begin{aligned} \sum_{i=0}^{\infty} \beta^i \left[b_1 \frac{\partial\left((1-L)y_{t+i}\right)^2}{\partial y_t} \right] &= 2b_1 \left[(1-L)y_t - \beta(1-L)\, y_{t+1} \right] \\ &= 2b_1 \left[(1-L)\, (y_t - \beta y_{t+1}) \right] \\ &= 2b_1 \left[(1-L)\, (1-\beta F)\, y_t \right] \end{aligned} \qquad \textbf{(A. 13)}$$

and for k=2

$$\begin{aligned} \sum_{i=0}^{\infty} \beta^i \left[b_2 \frac{\partial\left((1-L)^2 y_{t+i}\right)^2}{\partial y_t} \right] &= 2b_2 [(1-L)^2 y_t - 2\beta(1-L)^2 y_{t+1} + \beta^2 (1-L)^2 y_{t+2}] \\ &= 2b_2 [(1-L)^2 (y_t - 2\beta y_{t+1} + \beta^2 y_{t+2})] \\ &= 2b_2 [(1-L)^2 (1-\beta F)^2 y_t]. \end{aligned} \qquad \textbf{(A. 14)}$$

Continuing like this, we can derive the general expression for the derivative of the k^{th} term, which is

$$\sum_{i=0}^{\infty} \beta^i \left[b_k \frac{\partial \left((1-L)^k y_{t+i} \right)^2}{\partial y_t} \right] = 2b_k \left[(1-L)^k (1-\beta F)^k \; y_t \right]. \qquad \textbf{(A. 15)}$$

Substituting **(A. 15)** into **(A. 12)** gives us the first order condition, equation (3):

$$(y_t - y_t^*) \; + \; \sum_{k=1}^{m} b_k \; \left[(1-L)(1-\beta F) \right]^k \; y_t \; = \; 0 \qquad \textbf{(A. 16)}$$

which can be rewritten as

$$\left[1 \; + \; \sum_{k=1}^{m} b_k [(1-L)(1-\beta F)]^k \right] y_t \; = \; y_t^*. \qquad \textbf{(A. 17)}$$

The expression in square brackets is a “self-reciprocal” polynomial, that is, the coefficient on $(\beta F)^k$ is the same as the coefficient on L^k. To see this, first consider the expression $b_k[(1-L)(1-\beta F)]^k$ for $k = 1$

$$\begin{aligned} b_1(1-L)(1-\beta F) \; &= \; b_1(1-L-\beta F+\beta) \\ &= \; b_1(-L+(1+\beta)-\beta F). \end{aligned} \qquad \textbf{(A. 18)}$$

The coefficients on L and βF are identical (they both equal $-b_1$). Next, consider $k = 2$

$$\begin{aligned} b_2[(1-L)(1-\beta F)]^2 \; &= \; b_2[\; (1-L)(1-\beta F) \;] \; [\; (1-L)(1-\beta F) \;] \\ &= \; b_2[\; -L+(1+\beta)-\beta F \;] \; [\; -L+(1+\beta)-\beta F \;] \\ &= \; b_2[\; L^2 \; - \; 2(1+\beta)L \; + \; (1+4\beta+\beta^2) \; - \; 2(1+\beta)\beta F \; + \; (\beta F)^2 \;] \end{aligned} \qquad \textbf{(A. 19)}$$

The coefficients on L^2 and $(\beta F)^2$ are identical (they both equal b_2), as are the coefficients on L and βF (they equal $-2b_2(1+\beta)$). In a similar fashion, it is possible to show for all

$k = 1, \ldots, m,$ that $b_k[(1-L)(1-\beta F)]^k$ is a self-reciprocal polynomial. Because $\sum_{k=1}^{m} b_k[(1-L)(1-\beta F)]^k$ is a summation over self-reciprocal polynomials, and a summation over symmetric terms produces symmetric terms, it follows that $\sum_{k=1}^{m} b_k[(1-L)(1-\beta F)]^k$ is a self-reciprocal polynomial as well. The addition of a constant 1 does not affect the coefficients on lags or leads, so $1 + \sum_{k=1}^{m} b_k[(1-L)(1-\beta F)]^k$ is also self-reciprocal, as claimed. It can be written explicitly in self-reciprocal form as:

$$1 + \sum_{k=1}^{m} b_k[(1-L)(1-\beta F)]^k = C\,(L,\ \beta F), \qquad \textbf{(A. 20)}$$

where

$$C(L,\beta F) \equiv c_m L^{m} + \ldots + c_1 L + c_0 + c_1(\beta F) + \ldots + c_m(\beta F)^m\ . \qquad \textbf{(A. 21)}$$

Tinsley (1993) shows that a self-reciprocal polynomial with *m* lags and discounted leads can be factored as

$$C(L,\beta F) = \hat{A}(L)\hat{A}(\beta F) \qquad \textbf{(A. 22)}$$

where

$$\hat{A}(z) = \hat{\alpha}_0 + \hat{\alpha}_1 z + \hat{\alpha}_2 z^2 + \ldots + \hat{\alpha}_m z^{m} \qquad \textbf{(A. 23)}$$

and for $i = 0, 1, \ldots, m$

$$c_i = \sum_{j=i}^{m} \hat{\alpha}_{m+i-j}\ \hat{\alpha}_{m-j}\ \beta^{m-j}. \qquad \textbf{(A. 24)}$$

We can verify that this is true for a polynomial of degree two by expanding $\hat{A}(L)\hat{A}(\beta F)$ when $\hat{A}(z) = \hat{\alpha}_0 + \hat{\alpha}_1 z + \hat{\alpha}_2 z^2$. In this case,

$$\hat{A}(L)\hat{A}(\beta F) = \left(\hat{\alpha}_0 + \hat{\alpha}_1 L + \hat{\alpha}_2 L^2\right)\left(\hat{\alpha}_0 + \hat{\alpha}_1(\beta F) + \hat{\alpha}_2(\beta F)^2\right)$$
$$= \hat{\alpha}_0\hat{\alpha}_2 L^2 + (\hat{\alpha}_0\hat{\alpha}_1 + \hat{\alpha}_1\hat{\alpha}_2\beta)L + (\hat{\alpha}_0^2 + \hat{\alpha}_1^2\beta + \hat{\alpha}_2^2\beta^2) + (\hat{\alpha}_0\hat{\alpha}_1 + \hat{\alpha}_1\hat{\alpha}_2\beta)(\beta F) + \hat{\alpha}_0\hat{\alpha}_2(\beta F)^2 \qquad \textbf{(A. 25)}$$

Comparing this with **(A. 21)** indicates

$$\begin{aligned} c_2 &= \hat{\alpha}_0\hat{\alpha}_2 \\ c_1 &= \hat{\alpha}_0\hat{\alpha}_1 + \hat{\alpha}_1\hat{\alpha}_2\beta \\ c_0 &= \hat{\alpha}_0^2 + \hat{\alpha}_1^2\beta + \hat{\alpha}_2^2\beta^2 \end{aligned} \qquad \textbf{(A. 26)}$$

consistent with **(A. 24)**.

Given the results in **(A. 22)** to **(A. 24)**, we can factor equation **(A. 17)** as

$$\hat{A}(L)\ \hat{A}(\beta F)\ y_t = y_t^*. \qquad \textbf{(A. 27)}$$

The relationship between the cost parameters $b_1, \ldots, b_m$ and the polynomial coefficients $\hat{\alpha}_1, \ldots, \hat{\alpha}_m$ can be seen by expanding $1+\sum_{k=1}^{2} b_k[(1-L)(1-\beta F)]^k$ for $m = 2$. Using equations **(A. 18)** and **(A. 19)**, gives

$$1+\sum_{k=1}^{2} b_k[(1-L)(1-\beta F)]^k$$
$$= 1 + b_1[\ -L + (1+\beta) - \beta F\ ] + b_2\left[\ L^2 - 2(1+\beta)L + (1+4\beta+\beta^2) - 2(1+\beta)\beta F + (\beta F)^2\ \right]$$
$$= b_2 L^2 - \left[b_1 - 2(1+\beta)\right] L + \left[1 + b_1(1+\beta) + b_2(1+4\beta+\beta^2)\right] - \left[b_1 - 2(1+\beta)\right]\beta F + b_2(\beta F)^2 \qquad \textbf{(A. 28)}$$

A comparison of equations **(A. 21)**, **(A. 24)** and **(A. 28)** gives us

$$\begin{aligned} c_2 &= \hat{\alpha}_0\hat{\alpha}_2 &&= b_2 \\ c_1 &= \hat{\alpha}_0\hat{\alpha}_1 + \hat{\alpha}_1\hat{\alpha}_2\beta &&= -b_1 - 2b_2(1+\beta) \\ c_0 &= \hat{\alpha}_0^2 + \hat{\alpha}_1^2\beta + \hat{\alpha}_2^2\beta^2 &&= 1 + b_1(1+\beta) + b_2(1+4\beta+\beta^2) \end{aligned} \qquad \textbf{(A. 29)}$$

Given β, it is straightforward to obtain the cost parameters $b_1, \ldots, b_m$ from either $\hat{\alpha}_1, \ldots, \hat{\alpha}_m$ or $c_1, \ldots, c_m$. However, expressions for $\hat{\alpha}_1, \ldots, \hat{\alpha}_m$ are non-linear and complicated.

A first step in simplifying **(A. 27)** involves making the polynomial "monic", so the constant term is normalized to 1. Dividing both sides of **(A. 27)** by $\hat{\alpha}_0^2$ gives a new polynomial A, with coefficients α:

$$A(L)\ A(\beta F)\ y_t = \frac{y_t^*}{\hat{\alpha}_0^2} \qquad \textbf{(A. 30)}$$

where

$$\begin{aligned} A(z) &= 1 + \alpha_1 z + \ldots + \alpha_m z^m \\ \alpha_k &= \frac{\hat{\alpha}_k}{\hat{\alpha}_0} \end{aligned} \qquad \textbf{(A. 31)}$$

Let c represent the arbitrary constant, $1/\hat{\alpha}_0^2$. This can be fixed by assuming that the agent expects to reach the target path in a steady state; so that, when $L = F = 1$, then $y = y^*$. Then,

$$c = \frac{1}{\hat{\alpha}_0^2} = A(1)\ A(\beta) \qquad \textbf{(A. 32)}$$

which gives us equation (4),

$$A(L)\ A(\beta F)\ y_t = c y_t^* \qquad \textbf{(A. 33)}$$

This is an m-order difference equation, which can be simplified by expressing it as a first order difference equation in matrix form. As shorthand, let $A(L)y_t = x_t$. Then the left hand side of **(A. 33)** can be written as

$$A(\beta F)\ x_t = x_t + \alpha_1\beta x_{t+1} + \alpha_2\beta^2 x_{t+2} + \ldots + \alpha_m\beta^m x_{t+m} \tag{A. 34}$$

Put future terms in matrix form, with names of corresponding vectors written in bold underneath

$$\begin{aligned} A(\beta F)\ x_t &= \underset{}{x_t} + \underset{\boldsymbol{b}'}{\begin{bmatrix} \alpha_1\beta & \alpha_2\beta^2 & \ldots & \alpha_m\beta^m \end{bmatrix}} \underset{\boldsymbol{g}_{t+1}}{\begin{bmatrix} x_{t+1} \\ x_{t+2} \\ \vdots \\ x_{t+m} \end{bmatrix}} \\ &= x_t + \boldsymbol{b}'\ \boldsymbol{g}_{t+1} \end{aligned} \tag{A. 35}$$

Equating this with the right hand side of **(A. 33)** gives us

$$\begin{aligned} A(\beta F)\ x_t = x_t + \boldsymbol{b}'\boldsymbol{g}_{t+1} &= cy_t^* \\ x_t &= cy_t^* - \boldsymbol{b}'\boldsymbol{g}_{t+1} \end{aligned} \tag{A. 36}$$

Because x_t is the first element of the vector $\boldsymbol{g}_t$, this equation can be re-expressed as a first-order difference equation. Construct an *m*-vector of zero's, with 1 in the first row. That is, $\boldsymbol{\xi}_m' = [\ 1\ 0\ 0\ 0\ \ldots\ ]$. Then

$$\boldsymbol{\xi}_m'\ \boldsymbol{g}_t = x_t = cy_t^* - \boldsymbol{b}'\boldsymbol{g}_{t+1} \tag{A. 37}$$

To solve this, we first find an expression for $\boldsymbol{g}_t$. The following system has **(A. 37)** as its first equation, with other elements of $\boldsymbol{g}_t$ being explained by identities. Names of corresponding matrices are written in bold underneath.

$$\underset{\boldsymbol{g_t}}{\begin{bmatrix} x_t \\ x_{t+1} \\ \vdots \\ x_{t+m-2} \\ x_{t+m-1} \end{bmatrix}} = \underset{\boldsymbol{f_t}}{\begin{bmatrix} cy_t^* \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}} + \underset{\boldsymbol{G}}{\begin{bmatrix} -\alpha_1\beta & -\alpha_2\beta^2 & \dots & -\alpha_{m-1}\beta^{m-1} & -\alpha_m\beta^m \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}} \underset{\boldsymbol{g_{t+1}}}{\begin{bmatrix} x_{t+1} \\ x_{t+2} \\ \vdots \\ x_{t+m-1} \\ x_{t+m} \end{bmatrix}} \qquad \textbf{(A. 38)}$$

G is an *m*x*m* matrix comprising the vector $-\boldsymbol{b}'$ in the first row, an (*m*-1)x1 vector of zeros in the rest of the last column and an identity matrix, I_{m-1} of rank *m*-1 in the lower left corner. Using **(A. 38)** to substitute for successive leads of g_t, **(A. 37)** can be rewritten as an infinite forward sum:

$$\begin{aligned} x_t &= \boldsymbol{\xi}_m' \, \boldsymbol{g_t} \\ &= \boldsymbol{\xi}_m' \left[\boldsymbol{f_t} + \boldsymbol{G}\boldsymbol{g}_{t+1} \right] \\ &= \boldsymbol{\xi}_m' \left[\boldsymbol{f_t} + \boldsymbol{G}(\boldsymbol{f}_{t+1} + \boldsymbol{G}\boldsymbol{g}_{t+2}) \right] \\ &= \boldsymbol{\xi}_m' \left[\boldsymbol{f_t} + \boldsymbol{G}\boldsymbol{f}_{t+1} + \boldsymbol{G}^2(\boldsymbol{f}_{t+2} + \boldsymbol{G}\boldsymbol{g}_{t+3}) \right] \\ &= \boldsymbol{\xi}_m' \left[\boldsymbol{f_t} + \boldsymbol{G}\boldsymbol{f}_{t+1} + \boldsymbol{G}^2\boldsymbol{f}_{t+2} + \boldsymbol{G}^3(\boldsymbol{f}_{t+3} + \boldsymbol{G}\boldsymbol{g}_{t+4}) \right] \\ &= \boldsymbol{\xi}_m' \sum_{i\;0}^{\infty} \boldsymbol{G}^i \boldsymbol{f}_{t+i} \end{aligned} \qquad \textbf{(A. 39)}$$

where $G^0 = 1$. **(A. 39)** provides a useful shorthand to which we return, when we discuss estimation below. But to derive the decision rule, polynomial notation is more convenient. First, to put the expression back in terms of *y* and *y**, we can substitute back for $x_t = A(L)y_t$ and $f_{t+i} = \xi_m c y_{t+i}^*$:

$$A(L)\ y_t \ = \ \boldsymbol{\xi}_{\boldsymbol{m}}^{\prime} \sum_{\boldsymbol{i}=\boldsymbol{0}}^{\infty} \boldsymbol{G}^{\boldsymbol{i}}\ \boldsymbol{\xi}_{\boldsymbol{m}}\ c y_{t+i}^{*} \qquad \textbf{(A. 40)}$$

This replaces the forward operator in **(A. 33)** with an infinite forward sum.

To put **(A. 40)** in error correction form involves expressing both sides as a level term and a weighted sum of differences. The left hand side can be reexpressed as

$$A(L)y_t \ = \ \Delta y_t \ + \ A(1)\ y_{t-1} \ - \ A^{*}(L)\Delta y_{t-1} \qquad \textbf{(A. 41)}$$

where $A^{*}(L) \ = \ \alpha_1^{*} \ + \ \alpha_2^{*}L \ + \ \ldots \ + \ \alpha_{m-1}^{*}L^{m-2}$

and $\alpha_k^{*} \ = \ \sum_{j=k+1}^{m} \alpha_j$ for *k = 1, 2, ... , m-1*

To verify this, expand *A(L)* with *m* = 3:

$$\begin{aligned} A(L)y_t &= y_t + \alpha_1 y_{t-1} + \alpha_2 y_{t-2} + \alpha_3 y_{t-3} \\ &= y_t + \alpha_1 y_{t-1} + \alpha_2 y_{t-2} + (\alpha_3 y_{t-2} - \alpha_3 y_{t-2}) \ + \ \alpha_3 y_{t-3} \\ &= y_t + \alpha_1 y_{t-1} + (\alpha_2 + \alpha_3) y_{t-2} - \alpha_3 \Delta y_{t-2} \\ &= y_t + (\alpha_1 + \alpha_2 + \alpha_3) y_{t-1} - (\alpha_2 + \alpha_3)\Delta y_{t-1} - \alpha_3 \Delta y_{t-2} \qquad \textbf{(A. 42)} \\ &= y_t \ - \ y_{t-1} \ + \ (1 + \alpha_1 + \alpha_2 + \alpha_3) y_{t-1} - (\alpha_2 + \alpha_3)\Delta y_{t-1} - \alpha_3 \Delta y_{t-2} \\ &= y_t \ - \ y_{t-1} \ + \ (1 + \alpha_1 + \alpha_2 + \alpha_3) y_{t-1} - \alpha_1^{*} \Delta y_{t-1} - \alpha_2^{*} \Delta y_{t-2} \\ &= \Delta y_t \ + \ A(1) y_{t-1} - A^{*}(L)\ \Delta y_{t-1} \end{aligned}$$

with $\alpha_1^{*} = \alpha_2 + \alpha_3$ and $\alpha_2^{*} = \alpha_3$ as claimed.

Substituting **(A. 41)** into **(A. 40)** gives

$$\Delta y_t \ + \ A(1) y_{t-1} \ - \ A^{*}(L)\Delta y_{t-1} \ = \ \boldsymbol{\xi}_{\boldsymbol{m}}^{\prime} \sum_{\boldsymbol{i}=\boldsymbol{0}}^{\infty} \boldsymbol{G}^{\boldsymbol{i}}\ \boldsymbol{\xi}_{\boldsymbol{m}}\ c y_{t+i}^{*} \qquad \textbf{(A. 43)}$$

The next step is the decomposition of the right hand side into levels and changes. I first show this in scalar terms, which seems simpler and provides a representation that can be interpreted as a decision rule. I then present it in matrix algebra, which is more suited to estimation.

As pre and post multiplication by the selection vector $\boldsymbol{\xi_m}$ picks out the top left hand element of $\boldsymbol{G^i}$, the coefficients on future values of y^* are scalars. So, the right hand side of **(A. 43)** can be represented as a polynomial in the forward operator:

$$\boldsymbol{\xi_m'} \sum_{i=0}^{\infty} \boldsymbol{G^i}\, \boldsymbol{\xi_m}\, c y^*_{t+i} = \left(\tilde{d}_0 + \tilde{d}_1 F^1 + \tilde{d}_2 F^2 + \ldots + \tilde{d}_\infty F^\infty\right) y^*_t \qquad \textbf{(A. 44)}$$
$$= \tilde{D}(F)\, y^*_t$$

$\tilde{D}$ is an infinite order polynomial, with typical element

$$\tilde{d}_i = c\, \xi_m'\, G^i\, \xi_m \qquad \textbf{(A. 45)}$$

If $m = 2$, as is the case with many FRB/US equations, then

$$\boldsymbol{G} = \begin{bmatrix} -\alpha_1\beta & -\alpha_2\beta^2 \\ 1 & 0 \end{bmatrix} \qquad \textbf{(A. 46)}$$

and

$$\tilde{D}(F) = c\,[\,1 - \alpha_1\beta\, F + \left(\alpha_1^2 - \alpha_2\right)\beta^2 F^2$$
$$+ \left(\alpha_1^3 + 2\alpha_1\alpha_2\right)\beta^3 F^3 + \left(\alpha_1^4 - 3\alpha_1\alpha_2 + \alpha_2^2\right)\beta^4 F^4 + \ldots\,] \qquad \textbf{(A. 47)}$$

Substitute **(A. 44)** into **(A. 43)**. Then add and subtract $A(1)y^*_{t-1}$ and rearrange to make Δy a function of last period's 'disequilibrium' $(y^*_{t-1} - y_{t-1})$:

$$\begin{aligned} \Delta y_t &= \; - A(1)y_{t-1} \; + \; A^*(L)\Delta y_{t-1} \; + \; \tilde{D}(F)\; y_t^* \; + \left[A(1)y_{t-1}^* \; - \; A(1)y_{t-1}^*\right] \\ &= \; A(1)(y_{t-1}^* \; - \; y_{t-1}) \; + \; A^*(L)\Delta y_{t-1} \; + \; \tilde{D}(F)\; y_t^* \; - \; A(1)y_{t-1}^* \end{aligned} \qquad \textbf{(A. 48)}$$

Group and difference the coefficients on y* by defining

$$D(F)\Delta y_t^* \; = \; \tilde{D}(F)\; y_t^* - \; A(1)\; y_{t-1}^* \qquad \textbf{(A. 49)}$$

where

$$D(F)\; \Delta y_t^* \; = \left[\; d_0 \; + \; d_1 F \; + \; d_2 F^2 + \; ... \;\right] \Delta y_t^*. \qquad \textbf{(A. 50)}$$

with

$$d_0 \; = \; A(1) \; = \; 1 \; + \; \alpha_1 \; + \; \alpha_2 + ... \; + \alpha_m \qquad \textbf{(A. 51)}$$

and, for j = 1, 2, ... ∞

$$d_j \; = \; A(1) \; - \sum_{i\;0}^{j-1} \; \tilde{d}_i \qquad \textbf{(A. 52)}$$

To verify this, expand $\tilde{D}(F)\; y_t^* \; - \; A(1)y_{t-1}^*$ as

$$
\begin{aligned}
&\tilde{D}(F)y_t^* \; - \; A(1)y_{t-1}^* \\
&= \left(\tilde{d}_0+\tilde{d}_1F+\tilde{d}_2F^2+\tilde{d}_3F^3 \; + \; \ldots \right) y_t^* \; - \; A(1)y_{t-1}^* \; + \left[A(1)\; y_t^* \; - \; A(1)y_t^*\right] \\
&= A(1)\Delta y_t^* \; - \; A(1)\; y_t^* + \tilde{d}_0 y_t^* \; + \left[\tilde{d}_1F+\tilde{d}_2F^2+\tilde{d}_3F^3 \; + \; \ldots \right] y_t^* \\
&= A(1)\Delta y_t^* \; - \left[A(1)-\tilde{d}_0\right] y_t^* \; + \; \tilde{d}_1 y_{t+1}^* \; + \left[\tilde{d}_2F^2+\tilde{d}_3F^3 \; + \; \ldots \right] y_t^* \\
&= A(1)\Delta y_t^* \; + \left[A(1)-\tilde{d}_0\right] \Delta y_{t+1}^* \; - \left[A(1)-\tilde{d}_0-\tilde{d}_1\right] y_{t+1}^* + \left[\tilde{d}_2F^2+\tilde{d}_3F^3 \; + \; \ldots \right] y_t^* \\
&= A(1)\Delta y_t^* \; + \left[A(1)-\tilde{d}_0\right]\Delta y_{t+1}^* \; + \left[A(1)-\tilde{d}_0-\tilde{d}_1\right]\Delta y_{t+2}^* - \left[A(1)-\tilde{d}_0-\tilde{d}_1-\tilde{d}_2\right] y_{t+2}^* \; + \left[\tilde{d}_3F^3+\ldots\right]
\end{aligned}
\qquad \textbf{(A. 53)}
$$

From continuing the expansion of equation **(A. 53)** in this manner the coefficients on lags of Δy^* can be seen to match **(A. 52)**.

Substituting **(A. 49)** into **(A. 48)** then yields

$$\Delta y_t \; = \; A(1)(y_{t-1}^* - y_{t-1}) \; - \; A^*(L)\Delta y_{t-1} \; + \; D(F)\Delta y_t^* . \qquad \textbf{(A. 54)}$$

which can be rewritten as the decision rule (1)

$$\Delta y_t \; = \; a_0(y_{t-1}^* - y_{t-1}) \; + \; \sum_{k=1}^{m-1} a_k \Delta y_{t-k} \; + \; E_{t-1} \; \sum_{j=0}^{\infty} d_j \Delta y_{t+j}^* \qquad \textbf{(A. 55)}$$

Where, as shown above,

$$a_0 \; = \; d_0 \; = \; A(1) \; = \; 1+\alpha_1+\ldots+\alpha_m \; = \; 1+\sum_{j=1}^{m} \alpha_j$$

for k = 1, 2, ... , m $\quad \sum_{k=1}^{m-1} a_k \Delta y_{t-k} \; = \; -A^*(L)\Delta y_{t-1}$ where $a_k \; = \; -\alpha_k^* = \; -\sum_{j=k+1}^{m} \alpha_j$

and, for j = 1, 2, ... ∞

$$
\begin{aligned}
d_j \; &= \; A(1) \; - \; \sum_{i=0}^{j-1} \tilde{d}_i \\
&= \; A(1) \; - \; A(1)A(\beta)\sum_{i=0}^{j-1} \xi' G^i \xi
\end{aligned}
$$

A matrix representation

(A. 54) provides an equation that is simple to interpret, but cannot be estimated. One reason for this is that expected values of the target y^* are not observable. Another is that **(A. 54)** contains an infinite number of coefficients (albeit reflecting $m+n$ underlying parameters). We can address both these issues after expressing the forward leads in matrix algebra.

Return to **(A. 39)**, which is an infinite sum of leads of the vector $\boldsymbol{f}$. Each lead of $\boldsymbol{f}$ can be decomposed into the lagged level and a sum of changes. That is, $f_{t+i} = \sum_{j=0}^{i} \Delta f_{t+j} + f_{t-1}$. To see this, add and subtract lagged levels from the right hand side:

$$
\begin{aligned}
f_t &= f_t - f_{t-1} + f_{t-1} \\
&= \Delta f_t + f_{t-1} \\
f_{t+1} &= \Delta f_{t+1} + f_t \\
&= \Delta f_{t+1} + \Delta f_t + f_{t-1}
\end{aligned}
\qquad \textbf{(A. 56)}
$$

And so on. The weighted summation of leads of $\boldsymbol{f}$ can then be expressed as:

$$
\begin{aligned}
&\sum_{i=0}^{\infty} G^i f_{t+i} \\
&= G^0 f_t + G^1 f_{t+1} + \ldots + G^{\infty} f_{t+\infty} \\
&= G^0\left(\Delta f_t + f_{t-1}\right) + G^1\left(\Delta f_{t+1} + \Delta f_t + f_{t-1}\right) + \ldots + G^{\infty}\left(\sum_{j=0}^{\infty} \Delta f_{t+j} + f_{t-1}\right) \\
&= \sum_{j=0}^{\infty} G^j f_{t-1} + \sum_{j=0}^{\infty} G^j \Delta f_t + \sum_{j=1}^{\infty} G^j \Delta f_{t+1} + \ldots + \sum_{j=k}^{\infty} G^j \Delta f_{t+k} + \ldots \\
&= [I-G]^{-1} f_{t-1} + [I-G]^{-1} \Delta f_t + [I-G]^{-1} G^1 \Delta f_{t+1} + \ldots + [I-G]^{-1} G^k \Delta f_{t+k} + \ldots \\
&= [I-G]^{-1} f_{t-1} + \sum_{k=0}^{\infty} [I-G]^{-1} G^k \Delta f_{t+k}
\end{aligned}
\qquad \textbf{(A. 57)}
$$

where the second last line uses the formula for the sum of an infinite series $\sum_{j=k}^{\infty} G^{j} = [I-G]^{-1}G^{k}$ with $G^{0} = I$. Substituting **(A. 57)** into **(A. 39)** gives:

$$x_t = \xi_m^{/}\,[I-G]^{-1}f_{t-1} + \xi_m^{/}\sum_{k=0}^{\infty}[I-G]^{-1}G^{k}\Delta f_{t+k} \qquad \textbf{(A. 58)}$$

From the definition of $f_t = \xi_m c y_t^{*}$ we have:

$$\begin{aligned}\Delta f_t &= \xi_m c y_t^{*} - \xi_m c y_{t-1}^{*} \\ &= \xi_m c \Delta y_t^{*}\end{aligned} \qquad \textbf{(A. 59)}$$

Substituting these into **(A. 58)** and substituting **(A. 41)** for x_t gives an expression in terms of levels and differences of $y*$:

$$\Delta y_t + A(1)\,y_{t-1} - A^{*}(L)\Delta y_{t-1} = c\xi_m^{/}[I-G]^{-1}\xi_m y_{t-1}^{*} + c\xi_m^{/}\sum_{k=0}^{\infty}[I-G]^{-1}G^{k}\xi_m\Delta y_{t+k}^{*} \qquad \textbf{(A. 60)}$$

In a steady state, $y = y*$ and $\Delta y = \Delta y^{*} = 0$. This gives

$$A(1) = c\xi_m^{/}[I-G]^{-1}\xi_m \qquad \textbf{(A. 61)}$$

Substituting this into **(A. 60)** gives:

$$\begin{aligned}\Delta y_t + A(1)\,y_{t-1} - A^{*}(L)\Delta y_{t-1} &= A(1)y_{t-1}^{*} + c\xi_m^{/}\sum_{k=0}^{\infty}[I-G]^{-1}G^{k}\xi_m\Delta y_{t+k}^{*} \\ &= A(1)y_{t-1}^{*} + Z_t\end{aligned} \qquad \textbf{(A. 62)}$$

where Z_t represents the contribution from the expected growth of the target. We return to this formulation in considering model-consistent expectations, below. Rearranging gives:

$$\Delta y_t \;=\; A(1)\left(y^*_{t-1} \;-\; y_{t-1}\right) \;+\; A^*(L)\Delta y_{t-1} \;+\; Z_t \tag{A. 63}$$

This differs from **(A. 54)** in that a matrix expression, Z, is substituted for the forward polynomial D, which was defined recursively.

VAR Expectations

Suppose that the target, y^* can be identified outside the PAC equation. In the terminology used earlier, this implies that there is no "stationary component" of the target, so $y^* = y^{1*}$. Also suppose that expectations of y^* are generated as if by a vector-auto-regression (VAR). Let z be an n-vector of variables comprising the determinants of expectations of the target y^*. Define y^*_t as the first element of z_t, so:

$$y^*_t \;=\; \xi^{\prime}_n \, z_t \tag{A. 64}$$

where $\xi^{\prime}_n$ is a selection vector comprised of 1 as the first element then n-1 zeros. Then, define H as the n x n matrix of coefficients generating one-period ahead forecasts of z. That is:

$$z_{t+1} \;=\; Hz_t \tag{A. 65}$$

If z contains more than one lag of a variable the corresponding row of H will contain zeros except for a 1 on the relevant lag. H is often also restricted to be consistent with assumptions about homogeneity and orders of integration.
Future levels of z are obtained by repeated substitution:

$$z_{t+2} \;=\; Hz_{t+1} \;=\; H(Hz_t) \;=\; H^2 z_t \tag{A. 66}$$

more generally,

$$z_{t+i} = H^{i} z_t \tag{A. 67}$$

And, to obtain differences:

$$\begin{aligned} \Delta z_{t+i} &= H^{i} z_t - H^{i-1} z_t \\ &= (H - I)\, H^{i-1} z_t \end{aligned} \tag{A. 68}$$

This can be lagged an extra period, to put it in terms of predetermined variables:

$$\begin{aligned} \Delta z_{t+i} &= (H - I)\, H^{i-1} \left(H z_{t-1}\right) \\ &= (H - I)\, H^{i}\, z_{t-1} \\ &= H^{i} (H - I)\, z_{t-1} \end{aligned} \tag{A. 69}$$

An estimation equation

Let Z_t represent the contribution from the expected growth of the target, as in **(A. 62)**. That is,

$$Z_t = c\, \xi_m' \sum_{k=0}^{\infty} [I_m - G]^{-1} G^{k} \xi_m\, \Delta y^{*}_{t+k} \tag{A. 70}$$

As y^* is an element of the information vector z, (see equation **(A. 64)**), this can be re-expressed in terms of future levels of z :

$$Z_t = c\, \xi_m' \sum_{k=0}^{\infty} [I_m - G]^{-1} G^{k} \xi_m\, \xi_n'\, \Delta z_{t+k} \tag{A. 71}$$

and, using **(A. 69)**, in terms of lagged levels of z :

$$Z_t \;=\; c\ \xi_m' \sum_{k\ \ 0}^{\infty} [I_m - G]^{-1} G^{\,k} \xi_m\ \ \xi_n'\ H^k \left(H - \ I_n\right) z_{t-1}$$

$$=\; h'\ z_{t-1} \tag{A. 72}$$

h' is a 1 x n row vector. Its transpose, an n x 1 column vector, is

$$h = \; c \left[\ \xi_m' \sum_{k\ \ 0}^{\infty} [I_m - G]^{-1} G^{\,k} \xi_m\ \ \xi_n'\ H^k \left(H - \ I_n\right) \right]'$$

$$=\; c\ \left(H - \ I_n\right)' \left(\sum_{k\ \ 0}^{\infty} [I_m - G]^{-1} G^{\,k} \xi_m\ \ \xi_n'\ H^k \right)' \xi_m \tag{A. 73}$$

$$=\; c\ \left(H' - \ I_n\right) \sum_{k\ \ 0}^{\infty} (H')^k\ \ \xi_n\ \ \xi_m'\ \ (G')^k\ \left([I_m - G]^{-1}\right)'\ \xi_m$$

This can be simplified by use of column stacks. One property of column stacks is that the column stack of a column vector equals the vector itself. So $vec(h) = h$. Another is that the column stack of a product of three matrices is $vec(ABC)\ =\ (C' \otimes A) vec(B)$. Repeated application of these properties gives:

$$
\begin{aligned}
h &= [vec(h)] \\
&= vec\left[c \left(H' - I_n\right) \sum_{k=0}^{\infty} (H')^k \, \xi_n \, \xi_m' \, (G')^k \left([I_m - G]^{-1}\right)' \xi_m \right] \\
&= c \left(\xi_m' \, [I_m - G]^{-1}\right) \otimes \left(H' - I_n\right) \; vec\left[\sum_{k=0}^{\infty} (H')^k \, \xi_n \, \xi_m' \, (G')^k \right] \\
&= c \left(\xi_m' \, [I_m - G]^{-1}\right) \otimes \left(H' - I_n\right) \; \sum_{k=0}^{\infty} G^k \otimes (H')^k \; vec\left[\xi_n \, \xi_m' \right] \\
&= c \left(\xi_m' \, [I_m - G]^{-1}\right) \otimes \left(H' - I_n\right) \sum_{k=0}^{\infty} \left(G \otimes H'\right)^k \xi_m \otimes \xi_n \\
&= c \left(\xi_m' \, [I_m - G]^{-1}\right) \otimes \left(H' - I_n\right) \left[I_{nm} - G \otimes H' \right]^{-1} \xi_m \otimes \xi_n
\end{aligned}
\qquad \textbf{(A. 74)}
$$

This represents the infinite sums of the *H* and *G* matrices as a finite expression that can be calculated. Substituting **(A. 71)** and **(A. 61)** into **(A. 60)** and rearranging gives:

$$\Delta y_t = A(1)\left(y^*_{t-1} - y_{t-1}\right) + A^*(L)\Delta y_{t-1} + h' z_{t-1} \qquad \textbf{(A. 75)}$$

where *h* is constructed as in **(A. 74)**. As discussed in the text, this equation can be estimated both in restricted form (with the restrictions implied by the construction of the *G* and *H* matrices) and as a simple linear regression.

Alternative specifications of the target

The previous discussion has assumed the target, y^* is a known scalar. More usually, the target will be a linear combination of several variables, with weights that need to be estimated. For trending variables, it is possible to estimate these weights from a static regression - the first step in the Engle-Granger 2-step procedure. These estimates are inefficient, but super-consistent. This approach is intuitive, simple to program and facilitates the imposition of cross-equation restrictions. For stationary variables, 2-stage estimation is no longer super-consistent.

In FRB/US, targets commonly comprise a trending component, denoted y^{1*}, which has a coefficient of 1 and contains elements with weights that are constrained or estimated outside the PAC equation and a stationary component, denoted y^{0*} with a coefficient vector γ that is estimated within the PAC equation. That is, $y^{*} = \gamma y^{0*} + y^{1*}$. Similarly, the VAR forecasts above can be decomposed as $h^{/}z_{t-1} = \gamma h_0^{/}z_{t-1}^{0} + h_1^{/}z_{t-1}^{1}$. We separately identify the lagged level of the trending component as a variable beginning with a "Q", expected values of the trending component as a variable beginning with a "Z" and combine lagged and expected values of the stationary component as one variable beginning with a "Z". The algebra and computer code for these transformations is set out below.

Alternative computer code for the *h* vector

FRB/US codes expectations of the target in three different ways. Equation **(A. 74)** applies when the VAR is estimated with the level of *y** as the dependent variable, as in **(A. 64)**. Then the vector *h* is coded as the vector PV_COF in the file get_pv as follows:

```
PV_COF = SUMA1*SUMAB*W2*W1_DL*KRON(IG,IH)
where  W2 =  INVERSE{ EYENM-KRON[MAT_G,TRANSPOSE(MAT_H) ] }
and   W1_DL =  KRON{ TRANSPOSE(IG)*INVERSE(EYEM-MAT_G),
                     TRANSPOSE(MAT_H)- EYEN }
```

SUMA1 = *A(1)* and SUMAB = *A*(β) are scalars, whose product is *c*
IG and IH correspond to the selection vectors ξ_m and ξ_n
EYEN, EYEM and EYENM are identity matrices of order *n*, *m* and *n*x*m* respectively
MAT_G and MAT_H correspond to the matrixes *G* and *H*.

Notational correspondences are close but inexact. For example, the order of rows in MAT_G and IG is reversed in *G* and ξ_m and elements of MAT_G have the opposite sign to those in *G*.

In FRB/US it is common to impose the restriction that the trending component of the target is integrated of order 1. In such cases, the VAR is estimated with Δy^* as the dependent variable instead of y^*. This specification facilitates the imposition of other restrictions (for example, inflation neutrality) and could make inference easier (though the VAR's are not used for this purpose). Differencing the target involves replacing **(A. 64)** with:

$$\Delta y_t^* = \xi_n' z_t \qquad \textbf{(A. 76)}$$

Using **(A. 65)** and **(A. 67)**, which are unchanged, this implies:

$$\Delta y_{t+k}^* = \xi_n' z_{t+k} = \xi_n' H^{k+1} z_{t-1} \qquad \textbf{(A. 77)}$$

Substituting this into **(A. 70)** implies a modified form of **(A. 72)**

$$\begin{aligned} Z_t^1 &= c\, \xi_m' \sum_{k\ 0}^{\infty} [I_m - G]^{-1} G^k \xi_m\, \xi_n'\, H^{k+1}\, z_{t-1} \\ &= h^{1'}\, z_{t-1} \end{aligned} \qquad \textbf{(A. 78)}$$

which differs from **(A. 72)** through the disappearance of I_n. The same steps as above gives a new version of **(A. 74)**:

$$h^{1'} = c \left(\xi_m' [I_m - G]^{-1}\right) \otimes H' \left[I_{nm} - G \otimes H' \right]^{-1} \xi_m \otimes \xi_n \qquad \textbf{(A. 79)}$$

and W1_DL in the computer code is replaced with:

```
W1_DD =  KRON(TRANSPOSE(IG)*INVERSE(EYEM-MAT_G),
                    TRANSPOSE(MAT_H) )
```

Another variation is to combine in one term the present value of expected values of the target (in FRB/US notation, a "Z" variable), with its lagged level (a "Q" variable). This is a common approach for coding of stationary contributions to the target. It is also common to drop the restriction that the elasticity of y with respect to this component of the target is one. Supposing the coefficient on the stationary component of the target is γ means rewriting **(A. 39)** as:

$$x_t \;=\; \gamma\ \xi_m^{/} \sum_{i=0}^{\infty} G^{\,i}\, f_{t+i} \tag{A. 81}$$

Substituting the definitions of $\boldsymbol{x}$ and $\boldsymbol{f}$ into this gives:

$$A(L)y_t \;=\; \gamma\ \xi_m^{/} \sum_{i=0}^{\infty} G^{\,i}\ \xi_m c\ y_{t+i}^{0*} \tag{A. 82}$$

Letting y^{0*} be the first element of a vector z^0 and using this in a VAR as above implies

$$y_{t+i}^{0*} \;=\; \gamma\ \xi_n^{/} z_{t+i}^{0} \;=\; \gamma\ \xi_n^{/} H^{\,i+1} z_{t-1}^{0}$$

substituting this in, then using the same steps as above gives

$$
\begin{aligned}
A(L)y_t &= \gamma\ c\ \xi_m^{/} \sum_{i=0}^{\infty} G^{\,i}\ \xi_m\ \xi_n^{/}\ H^{\,i+1}\ z_{t-1}^{0} \\
&= \gamma\ c \left[\left(\sum_{i=0}^{\infty} G^{\,i}\ \xi_m\ \xi_n^{/}\ H^{\,i+1} \right)^{/} \xi_m \right]^{/} z_{t-1}^{0} \\
&= \gamma\ c \left[\sum_{i=0}^{\infty} H^{/} H^{i^{/}}\ \xi_n\ \xi_m^{/}\ G^{\,i^{/}}\ \xi_m \right]^{/} z_{t-1}^{0} \\
&= \gamma\ c \left[vec\left(H^{/} \sum_{i=0}^{\infty} H^{i^{/}}\ \xi_n\ \xi_m^{/}\ G^{\,i^{/}}\ \xi_m \right) \right]^{/} z_{t-1}^{0} \\
&= \gamma\ c \left[\left(\xi_m^{/} \otimes H^{/} \right) vec\left(\sum_{i=0}^{\infty} H^{i^{/}}\ \xi_n\ \xi_m^{/}\ G^{\,i^{/}} \right) \right]^{/} z_{t-1}^{0} \qquad \textbf{(A. 84)} \\
&= \gamma\ c \left[\left(\xi_m^{/} \otimes H^{/} \right) \sum_{i=0}^{\infty} vec \left(H^{i^{/}}\ \xi_n\ \xi_m^{/}\ G^{\,i^{/}} \right) \right]^{/} z_{t-1}^{0} \\
&= \gamma\ c \left[\left(\xi_m^{/} \otimes H^{/} \right) \sum_{i=0}^{\infty} \left(G^{\,i} \otimes H^{i^{/}} \right) vec \left(\xi_n\ \xi_m^{/} \right) \right]^{/} z_{t-1}^{0} \\
&= \gamma\ c \left[\left(\xi_m^{/} \otimes H^{/} \right) \left(I_{nm} - G \otimes H^{/} \right)^{-1} \left(\xi_m \otimes \xi_n \right) \right]^{/} z_{t-1}^{0} \\
&= \gamma\ h^{0^{/}}\ z_{t-1}^{0}
\end{aligned}
$$

and W1_DL in the computer code is replaced with:

```
W1_LL =   KRON( TRANSPOSE(IG), TRANSPOSE(MAT_H) )
```

Model consistent expectations

To find an expected path for the target, y^* that is consistent with model forecasts, first substitute **(A. 41)** into **(A. 62)**:

$$
A(L)y_t = A(1)y_{t-1}^{*} + Z_t \qquad \textbf{(A. 86)}
$$

Multiply though by $A(\beta F)$

$$A(\beta F)A(L)y_t = A(\beta F)A(1)y^*_{t-1} + A(\beta F)Z_t \qquad \textbf{(A. 87)}$$

The left hand side can be put in terms of $y*$ by using **(A. 32)** and **(A. 33)**

$$A(1)A(\beta)y^*_t = A(\beta F)A(1)y^*_{t-1} + A(\beta F)Z_t \qquad \textbf{(A. 88)}$$

Then rearranging and expanding the polynomials gives

$$\begin{aligned} A(\beta F)Z_t &= A(1)\left[A(\beta)y^*_t - A(\beta F)y^*_{t-1} \right] \\ Z_t + \sum_{i=1}^{m} \alpha_i \beta^i Z_{t+i} &= A(1)\left[\left(1 + \sum_{i=1}^{m} \alpha_i \beta^i\right) y^*_t - \left(y^*_{t-1} + \sum_{i=1}^{m} \alpha_i \beta^i y^*_{t-1+i} \right) \right] \\ Z_t &= \sum_{i=1}^{m} \alpha_i \beta^i Z_{t+i} + A(1)\left[\Delta y^*_t + \sum_{i=1}^{m} \alpha_i \beta^i \left(y^*_t - y^*_{t-1+i} \right) \right] \end{aligned} \qquad \textbf{(A. 89)}$$

The last term is a weighted sum of higher order differences. This can be rewritten as a reweighted sum of first differences. To see this add and subtract intermediate levels of y^*_{t+k}:

$$\begin{aligned} y^*_{t+k} - y^*_t &= y^*_{t+k} - y^*_{t+k-1} + y^*_{t+k-1} + \dots - y^*_t \\ &= \sum_{i=0}^{k} \Delta y^*_{t+i} \end{aligned} \qquad \textbf{(A. 90)}$$

It follows that

$$
\begin{aligned}
\sum_{i=1}^{m} \alpha_i \beta^i \left(y_t^* - y_{t-1+i}^* \right) = {} & \alpha_1 \beta^1 0 + \alpha_2 \beta^2 \left(y_t^* - y_{t+1}^* \right) + \alpha_3 \beta^3 \left(y_t^* - y_{t+2}^* \right) + \alpha_4 \beta^4 \left(y_t^* - y_{t+3}^* \right) \\
& + \ldots + \alpha_m \beta^m \left(y_t^* - y_{t+m-1}^* \right) \\
= {} & - \alpha_2 \beta^2 \Delta y_{t+1}^* - \alpha_3 \beta^3 \left(\Delta y_{t+1}^* + \Delta y_{t+2}^* \right) - \ldots - \alpha_m \beta^m \sum_{k=0}^{m-1} \Delta y_{t+k}^* \qquad \textbf{(A. 91)} \\
= {} & - \sum_{k=1}^{m-1} \left(\sum_{j=k}^{m-1} \alpha_{j+1} \beta^{j+1} \right) \Delta y_{t+k}^*
\end{aligned}
$$

Substituting this into **(A. 87)** gives

$$
Z_t = \sum_{i=1}^{m} \alpha_i \beta^i Z_{t+i} + A(1) \left[\Delta y_t^* - \sum_{k=1}^{m-1} \left(\sum_{j=k}^{m-1} \alpha_{j+1} \beta^{j+1} \right) \Delta y_{t+k}^* \right] \qquad \textbf{(A. 92)}
$$

As discussed in the text, given Δy_{t+k}^* for $k = 1, \ldots T$ and Z_{t+j} for $j = T, \ldots T+m$, then this can be used to solve for Z_t.

References

J. Kennan, (1979) “The Estimation of Partial Adjustment Models with Rational Expectations”, *Econometrica*, Volume 47, Issue 6, pp. 1441-55

S. Nickell (1984) “Error Correction, Partial Adjustment and All that: An Expository Note” *Oxford Bulletin of Economics and Statistics*, 47, 2 119-29

J. J. Rotemberg (1982) “Price Adjustment and Aggregate Output” *The Review of Economic Studies* Volume 49, Issue 4 (Oct 1982), 517-531

P.A. Tinsley (1993) “Fitting both data and theories: Polynomial adjustment costs and error-correction decision rules”. Federal Reserve Board of Governors, *Finance and Economics Discussion Series*, 1993-21