PAC Basics

March 2014

The adjustment dynamics of many key nonfinancial variables in FRB/US are based on the generalized model of adjustment costs known as PAC, which stands for "Polynomial Adjustment Costs" (Tinsley, 1993, 2002). PAC is used in equations for consumption (ECO, ECD), fixed investment (EH, EPD, EPI, EPS), hours (LHP), and dividends (YNIDN). Section 1 outlines the derivation of the PAC decision rule. Section 2 describes an adjustment that makes the PAC equilibrium independent of the real rate of growth. Section 3 discusses how some of the FRB/US PAC equations are modified to account for the presence of some consumers and firms who do not optimize. Section 4 describes the estimation of PAC equations. The explicit expectations terms in each PAC equation can, in FRB/US simulations, have either model consistent solutions or so-called VAR solutions. Sections 5 and 6 present the formulas associated with the two expectations options.

This note focuses on the main aspects of PAC. Detailed derivations of the various equations and results that are presented here can be found in Tinsley (1993, 2002) and Brayton, Davis, and Tulip (BDT, 2000).

1. The PAC decision rule

Several observationally-equivalent cost functions can be used to derive the PAC specification. One such cost function, C_t,

$$C_t = \sum_{i=0}^{\infty} \beta^i [(y_{t+i} - y^*_{t+i})^2 + \sum_{k=1}^{m} b_k ((1-L)^k y_{t+i})^2], \tag{1}$$

penalizes both deviations of a variable y from its desired value y^*—as determined by an equilibrium condition—and movements in the level and $m-1$ time derivatives of the variable y. β (= .98) is a discount factor, and $b_k, k = 1, \ldots, m$, are cost parameters. (In this and all subsequent equations, future-dated variables should be interpreted as expected values.) A special case of the cost function is that of quadratic adjustment costs on changes in the *level* of y ($m = 1$). The terms equilibrium, target, and desired value will be used interchangeably to describe y^*. Minimization of costs yields the Euler equation,

$$(y_t - y^*_t) + \sum_{k=1}^{m} b_k [(1-L)(1-\beta F)]^k y_t = 0, \tag{2}$$

where L is the lag operator and F ($\equiv L^{-1}$) is the lead operator. This expression can be written more compactly as

$$A(\beta F)A(L)y_t - cy_t^* = 0, \tag{3}$$

where A is a polynomial in the lag and lead operators of order m, e.g., $A(L) = 1 + \alpha_1 L + \ldots + \alpha_m L^m$ and $A(\beta F) = 1 + \alpha_1 \beta F + \ldots + \alpha_m \beta^m F^m$, and $c = A(1)A(\beta)$ is a constant. The m parameters in A are transformations of the m cost parameters in equation (1).

The PAC decision rule is obtained by multiplication of both sides of equation (3) by $A(\beta F)^{-1}$ and some futher manipulation.

$$\Delta y_t = a_0(y_{t-1}^{1*} - y_{t-1}) + \sum_{i=1}^{m-1} a_i \Delta y_{t-i} + \sum_{i=0}^{\infty} d_i \Delta y_{t+i}^{1*} + \sum_{i=0}^{\infty} h_i y_{t+i}^{0*} \tag{4}$$

The derivation of equation (4) decomposes y^* into stationary I(0) (y^{0*}) and non-stationary I(1) (y^{1*}) components.[1] As can be seen, Δy responds to the lagged gap between the level of y and the nonstationary component of its equilibrium value, to lagged values of Δy, and to expected future values of Δy^{1*} and y^{0*}.[2] $m-1$ lags of the dependent variable enter because of higher-order adjustment frictions that are absent in the quadratic-cost specification. The forward weights, d_i and h_i, are nonlinear functions of β and the parameters of the polynomial A.

When PAC equations are placed in FRB/US PAC, the expected future sums of Δy^{1*} and y^{0*} are assigned to explicit variables, which are denoted as the expectations variables Z^1 and Z^0 in the following equation.

$$\Delta y_t = a_0(y_{t-1}^{1*} - y_{t-1}) + \sum_{i=1}^{m-1} a_i \Delta y_{t-i} + Z_t^1 + Z_t^0 \tag{5}$$

In simulations, the solutions for Z^1 and Z^0 may either be model-consistent or VAR-based. Sections 5 and 6 present the formulas for Z^1 and Z^0 under the two expectations alternatives.

2. Growth neutrality

The PAC equations in FRB/US include an adjustment to ensure that $y = y^{1*}$ in equilibrium, irrespective of the rate of growth of y. Assume that the economy is in a balanced growth equilibrium in which g is the rate of growth: $\Delta y = \Delta y^{1*} = \Delta x = g$, and $y^{0*} = 0$. Substitution of these values into equation (4) yields

$$(y^{1*} - y) = [1 - \sum_{i=1}^{m-1} a_i - \sum_{i=0}^{\infty} d_i]g \tag{6}$$

[1] Because the PAC framework is applied in FRB/US only to variables that are non-stationary, y^* always has a non-stationary component; a stationary component is present only in a subset of the PAC equations.

[2] The $\{a_i\}$ coefficients are related to the $\{\alpha_i\}$ parameters of A as follows: $a_0 = A(1)$; for $i > 0$, $a_i = \sum_{j=i+1}^{m} \alpha_i$.

As can be seen from this expression, $y = y^{1*}$ only if $\sum_{i=1}^{m-1} a_i + \sum_{i=0}^{\infty} d_i = 1$. While it is posssible to satisfy this condition by placing a restriction on the $\{a_i\}$ coefficients – recall that the d_i coefficients are functions of the a_i and the discount factor β – the restriction has the unwelcome interpretation that it sets the cost parameter in equation (1) associated with the first difference of the decision variable, b_1, to zero. Rather than following this approach, the FRB/US equations add a correction factor to the PAC decision rule on the view that the rule is an approximation to an underlying model of behavior around a growth equilibrium. The added term is,

$$[1 - \sum_{i=1}^{m-1} a_i - \sum_{i=0}^{\infty} d_i]g \tag{7}$$

where g is measured as the growth rate of potential output.[3] This term enters as an adjustment to the definition of Z^1.

3. Combining optimizing and nonoptimizing behavior

Most of the equations in FRB/US that use PAC to characterize the behavior of optimizing agents also assume that there are other agents who follow simple rules of thumb or face liquidity constraints.[4] Notably, the spending of some consumers moves in tandem with income and the investment of some firms is proportional to output. The spending or investment of this second group of agents is specified as $\Delta y = \Delta x$, where x denotes (the log of) income or output. Under the assumption that the spending or investment shares of the two types of agents are (approximately) constant over time, aggregation yields the following equation,

$$\Delta y_t = \gamma \left(a_0(y^{1*}_{t-1} - y_{t-1}) + \sum_{i=1}^{m-1} a_i \Delta y_{t-i} + Z^1_t + Z^0_t \right) + (1-\gamma)\Delta x_t \tag{8}$$

where γ is the fraction of consumption or investment spending that is the result of optimization.

4. Estimation

FRB/US has too many equations and coefficients for the whole model to be estimated simultaneously. As a result, most of the model is estimated one equation at a time. The approach to estimating each PAC equation involves a small model that combines the PAC equation with a a non-structural VAR that provides the projections needed to form the Z^1 and Z^0 expectations terms.

[3]The code that imposes the restriction makes use of the condition $\sum_{i=0}^{\infty} d_i = A(1)A(\beta F)\iota'_m[I_m - G]^{-2}]\iota_m$. The definitions of I_m and G are given below in section 6.

[4]The PAC equation for economy-wide hours, *lhp*, also assumes that there are two types of agents, but in this case the distinction is between fast-adjusting and slow-adjusting firms, not optimizers and non-optimizers.

Estimation takes place in three steps. In the first, observations on the nonstationary part of the target variable, y^{1*}, are created from either a steady-state equilibrium condition or a cointegration regression. Any stationary part of the target, y^{0*}, is assumed to be a function of observable macro variables like the output gap. The second step estimates the $\{\Gamma_i\}$ coefficients of the VAR in the small model, which has the general form,

$$X_t = \sum_{i=1}^{4} \Gamma_i X_{t-i} \tag{9}$$

in which X is a vector of variables that includes y^{0*} and Δy^{1*}.

In order to describe the third step, which estimates the $\{a_i\}$ coefficients of the PAC equation, it is convenient to rewrite the VAR in first-order companion form,

$$z_t = H z_{t-1} \tag{10}$$

$z_t' = [X_t', X_{t-1}', X_{t-2}', X_{t-3}']$ and H combines the $\{\Gamma_i\}$ coefficient matrices. Next, note that the PAC expectations terms can be represented as products of coefficient vectors and the VAR variables,

$$Z_t^0 = h_0' z_{t-1} \tag{11}$$

$$Z_t^1 = h_1' z_{t-1} \tag{12}$$

where the coefficient vectors h_0 and h_1 are themselves nonlinear functions of the PAC coefficients $\{a_i\}$, along with the fixed discount factor and the VAR coefficients H.[5] Because of the nonlinear dependence of the expectations terms on the PAC coefficients, the PAC equation is nonlinear in parameters. Parameter estimates are obtained with an iterative OLS approach. Initial values of the PAC coefficents are chosen and used to create estimates of Z^0 and Z^1. Given these estimates of the expectations terms, estimates of the PAC coefficients that directly appear in the equation are obtained by OLS. New estimates of the expectations variables are then formed and the PAC coefficients reestimated. Iterations continue until convergence.

5. Model consistent expectations formulas

The PAC expectations terms are weighted averages over the infinite horizon of the components of y^*,

[5] For more information on the structure of the VARs and the explicit definitions of h_0 and h_1, see the *VAR Expectations Basics* document.

$$Z_t^0 = \sum_{i=0}^{\infty} h_i y_{t+i}^{0*} \tag{13}$$

$$Z_t^1 = \sum_{i=0}^{\infty} d_i \Delta y_{t+i}^{1*} \tag{14}$$

The MCE formulas for Z^0 and Z^1 collapse the infinite-lead form of these expressions into a finite-lead form that is much more convenient to use. The derivation starts by multiplying both sides of the Euler equation (3) by $A(\beta F)^{-1}$.

$$A(L)y_t = A(\beta F)^{-1}A(\beta)A(1)y_t^{0*} + A(\beta F)^{-1}A(\beta)A(1)y_t^{1*} \tag{15}$$

The general form of this equation differs from the PAC decision rule only in that all of the terms involving y are on its left hand side. Consequently, the first term on the right hand side, which is the only one involving y^{0*}, must be Z^0.

$$Z^0 = A(\beta)A(1)A(\beta F)^{-1}y_t^{0*} \tag{16}$$

$$A(\beta F)Z^0 = A(\beta)A(1)y_t^{0*} \tag{17}$$

$$Z^0 = -\sum_{i=1}^{m} \alpha_i B^i Z_{t+i}^0 + A(B)A(1)y_t^{0*} \tag{18}$$

Equation 18 is the finite-lead form of the expression for the MC expectation of the stationary component of the PAC target variable.

The derivation of the finite-lead MCE formula for the non-stationary expectations term starts from the observation that the second term on the right hand side of equation (15) must be $Z_t^1 + A(1)y_{t-1}^{1*}$. Thus,

$$Z_t^1 = -A(1)y_{t-1}^{1*} + A(\beta)A(1)A(\beta F)^{-1}y_t^{1*} \tag{19}$$

$$A(\beta F)Z_t^1 = A(1)[A(\beta)y_t^{1*} - A(\beta F)y_{t-1}^{1*}] \tag{20}$$

$$Z_t^1 = -\sum_{i=1}^{m} \alpha_i \beta^i Z_{t+i}^1 + A(1)[\Delta y_t^{1*} + \sum_{k=1}^{m-1}(\sum_{j=k}^{m-1} \alpha_{j+1}\beta^{j+1})\Delta y_{t+k}^{1*}] \tag{21}$$

Equation 21 is the finite-lead model-consistent formula for the expectation of the non-stationary component of the PAC target variable.

6. VAR expectations formulas

The derivation of the VAR expectations formulas also starts from equations 13-14, which show the PAC expectations terms to be weighted averages over the infinite horizon of the components of y^*. The next step is to examine the weights, which are the following functions of the Euler equation coefficients and the discount factor,

$$
\begin{aligned}
h_i &= A(1)A(\beta)\iota'_m G^i \iota_m && (22)\\
d_i &= A(1)A(\beta)\iota'_m [I-G]^{-1} G^i \iota_m && (23)
\end{aligned}
$$

Here, ι_m is a $m \times 1$ vector with a one in the mth element and zeroes elsewhere, and G is the $m \times m$ matrix

$$
G = \begin{bmatrix} 0 & & I_{m-1} & \\ -\alpha_m \beta^m & -\alpha_{m-1}\beta^{m-1} & \dots & -a_1\beta \end{bmatrix}.
$$

Recall the companion form of the VAR system, $z_t = Hz_{t-1}$. Let y_t^{0*} be element $h0$ of z_t and Δy^{1*} be element $h1$, and define ι_{h0} and ι_{h1} as the corresponding selector vectors. Using this notation, each future expectation is a linear function of the VAR coefficients, H, and the VAR variables, z.

$$
\begin{aligned}
E_{t-1} y_{t+i}^{0*} &= \iota'_{h0} H^i z_{t-1} && (24)\\
E_{t-1} \Delta y_{t+i}^{1*} &= \iota'_{h1} H^i z_{t-1}. && (25)
\end{aligned}
$$

Some additional matrix algebra yields the the VAR expectations equations,

$$
\begin{aligned}
Z_t^0 &= h'_0 z_{t-1} && (26)\\
Z_t^1 &= h'_1 z_{t-1} && (27)
\end{aligned}
$$

whose coefficient vectors h_0 and h_1 are defined as follows.

$$
\begin{aligned}
h_0 &= A(1)A(\beta)[(\iota'_m I_m) \otimes H'][I_{nm} - (G \otimes H')]^{-1}[\iota_m \otimes \iota_{h0}] && (28)\\
h_1 &= A(1)A(\beta)[(\iota'_m (I_m - G)^{-1}) \otimes H'][I_{nm} - (G \otimes H')]^{-1}[\iota_m \otimes \iota_{h1}] && (29)
\end{aligned}
$$

References

Brayton, Flint, Morris Davis and Peter Tulip. 2000, “Polynomial Adjustment Costs in FRB/US,” unpublished manuscript.

Tinsley, Peter. 1993, “Fitting Both Data and Theories: Polynomial Adjustment Costs and Error-Correction Decision Rules,” *Finance and Economics Discussion Series*, 1993-21.

Tinsley, Peter. 2002, “Rational Error Correction,” *Computational Economics*, 19, 197-225. (Earlier version available as *Finance and Economics Discussion Series*, 1998-37.)