Cross spectral analysis - Eviews add-in

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Abstract

This document describes the use of cross spectral analysis techniques. Its objective is to determine the relationship between sinusoidal components of a pair of time series. The article alse describes an Eviews add-in to estimate cross spectral functions by using Monte Carlo simulations. It is recommended to review Wei (2006) for a more detailed description of the theory.

1 Cross spectral analysis theory

The cross spectrum of a pair of time series x_t and y_t is defined as the Fourier transform of its cross covariance function, for this it is required to define the covariance generating function such that:

$$R_{xy}(L) = \sum_{k=-\infty}^{\infty} R_{xy}(k)L^k$$

Where $R_{xy}(k)$ represents the cross covariance function between x_t and y_{t+k} and L is the lag operator. Therefore, its crossed spectrum is given by:

$$f_{xy}(w) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} R_{xy}(k)e^{-iwk} = \frac{1}{2\pi}R_{xy}(e^{-iwk})$$

Where e^{-iwk} is Euler's equation $e^{-iwk} = Cos(wk) - isen(wk)$. In the same way $f_{xx}(w)$ and $f_{yy}(w)$ correspond to the spectra of x_t and y_t respectively. Equivalently:

$$f_{xy}(w) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} R_{xy}(k)(Cos(wk) - isin(wk)) = c_{xy}(w) - iq_{xy}(w)$$

Where:

$$c_{xy}(w) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} R_{xy}(k)Cos(wk)$$

$$q_{xy}(w) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} R_{xy}(k)Sin(wk)$$

The function $c_{xy}(w)$ is defined as the cospectrum, the real component of $f_{xy}(w)$ and $q_{xy}(w)$ is the quadrature spectrum, the imaginary component of $f_{xy}(w)$. Alternatively, the cross spectrum can be expressed in its polar form as:

$$f_{xy}(w) = A_{xy}(w)e^{i\phi_{xy}(w)}$$

Where:

$$A_{xy}(w) = (c_{xy}(w)^2 + q_{xy}(w)^2)^{1/2}$$

$$\phi_{xy}(w) = tan^{-1}(\frac{-q_{xy}(w)}{c_{xy}(w)})$$

This can be seen by $w_t = \alpha Cos(wt) + \beta Sen(wt) = \rho Cos(wt + \theta)$ where $\rho = (\alpha^2 + \beta^2)^{1/2}$ is the amplitude of w_t and $\theta = tan^{-1}(-\beta/\alpha)$ is the phase angle of w_t. In this way the functions $A_{xy}(w)$ and $\phi_{xy}(w)$ are known as the cross amplitude spectrum and phase spectrum respectively. The amplitude spectrum can be interpreted as the covariance between the frequency component w of x_t and the frequency component w of y_t. The phase spectrum can be expressed in time units as $\tau = \phi_{xy}(w)/w$, this is known as the phase shift spectrum. The phase spectrum represents the average phase shift between x_t and y_t, that is, if one time series leads the other. If $\phi_{xy}(w)$ is negative, it is said that the frequency component w of x_t leads the frequency component w of y_t, that is, x_t anticipates the movements of y_t at that frequency. Additionally, there are other functions that help to interpret the estimates of the cross spectrum. The gain function and the square coherence function are defined as:

$$G_{xy}(w) = \frac{A_{xy}(w)}{f_x(w)}$$

$$K^2_{xy}(w) = \frac{|f_{xy}(w)|}{f_x(w)f_y(w)}$$

In practice the cross spectrum must be explained by several functions. The functions of the corss spectrum will be estimated using a frequentist approach. Let x_t and y_t be two time series, their univariate spectra can be estimated as:

$$\hat{f}_x(w) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} W_x(k)\hat{R}_x(k)e^{-iwk}$$

$$\hat{f}_y(w) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} W_y(k)\hat{R}_y(k)e^{-iwk}$$

Where $\hat{R_x}(k)$ is the estimator of the autocovariance of the series x_t of order k, this is calculated as: $\hat{R_x}(k) = \frac{1}{T}\sum_{t=1+k}^{T}(x_t - \overline{x})(x_{t-k} - \overline{x})$. In the same way $\hat{R_y}(k) = \frac{1}{T}\sum_{t=1+k}^{T}(y_t - \overline{y})(y_{t-k} - \overline{y})$ is the autocovariance function of y_t of order k. Now, the functions $W_x(k)$ and $W_y(k)$ are known as lag windows, which are used to give less weight to the autocovariance coefficients estimated with less precision. The bias of the estimators of the autocovariance function depends on k, in general, the higher values of k the greater the probability of a bias. However, the estimator is consistent.. The terms M_x and M_y are known as truncation points. In practice it is usual to use the functions expressed in table 1 and those are used in the add-in.

Window name	Lag window
Hamming	$\lambda(k) = 1 - 2*0.23 + 2*0.23cos(\frac{\pi k}{M})$ for $\mid k \mid\leq M$ and 0 otherwise
Hann	$\lambda(k) = 1 - 2*0.25 + 2*0.25cos(\frac{\pi k}{M})$ for $\mid k \mid\leq M$ and 0 otherwise
Bartlett	$\lambda(k) = 1 - \frac{\mid k\mid}{M}$ for $\mid k \mid\leq M$ and $\lambda(k) = 0$ for $\mid k \mid > M$
Parzen	$\lambda(k) = 1 - 6(\frac{K}{M})^2 + 6(\frac{\mid k\mid}{M})^3$ for $\mid k \mid\leq \frac{M}{2}$ and $\lambda(k) = 2(1-\frac{\mid k\mid}{M})^3$ for $\frac{M}{2}\leq\mid k \mid\leq M$
Rectangular	$\lambda(k) = 1$ for $\mid k \mid\leq M$ and $\lambda(k) = 0$ for $\mid k \mid > M$

Table 1: Lag windows.

Similarly, the cross spectrum between the two time series can be estimated as follows:

$$\hat{f_{xy}}(w) = \frac{1}{2\pi}\sum_{k=-M_{xy}}^{M_{xy}} W_{xy}(k)\hat{R_{xy}}(k)e^{-iwk}$$

Where $\hat{R_{xy}}(k)$ is the cross-covariance function, which, unlike the autocovariance function, is not symmetric around k. The function $W_{xy}(k)$ is the lag window and M_{xy} is the truncation point. The cross spectrum can also be written as:

$$\hat{f_{xy}}(w) = \hat{c_{xy}}(w) - i\hat{q_{xy}}(w)$$

Where:

$$\hat{c_{xy}}(w) = \frac{1}{2\pi}\sum_{k=-M_{xy}}^{M_{xy}} W_{xy}(k)\hat{R_{xy}}(k)Cos(wk) = \frac{1}{2\pi}(W_{xy}(0)\hat{R_{xy}}(0) + \sum_{k=1}^{M_{xy}} W_{xy}(k)(\hat{R_{xy}}(k) + \hat{R_{xy}}(-k))Cos(wk)$$

$$\hat{q_{xy}}(w) = \frac{1}{2\pi}\sum_{k=-M_{xy}}^{M_{xy}} W_{xy}(k)\hat{R_{xy}}(k)Sin(wk) = \frac{1}{2\pi}\sum_{k=1}^{M_{xy}} W_{xy}(k)(\hat{R_{xy}}(k) + \hat{R_{xy}}(-k))Sin(wk)$$

The previous equations are the estimators of the cospectrum and the quadrature spectrum respectively. In theory the truncation points and lag windows may differ to estimate the value of the univariate or crossed spectra for a set of time series, however, for simplicity the same truncation point and lag window is used in the add-in for all estimators. Once the cospectrum and the quadrature spectrum are calculated, the cross amplitude spectrum phase, phase shift, gain and coherence functions can be estimated as follows:

$$\hat{A_{xy}}(w) = (\hat{c_{xy}}(w)^2 + \hat{q_{xy}}(w)^2)^{1/2}$$

$$\hat{\phi_{xy}}(w) = tan^{-1}(\frac{-\hat{q_{xy}}(w)}{\hat{c_{xy}}(w)})$$

$$\hat{G_{xy}}(w) = \frac{\hat{A_{xy}}(w)}{\hat{f_x}(w)}$$

$$\hat{K^2_{xy}}(w) = \frac{|\hat{f_{xy}}(w)|}{\hat{f_x}(w)\hat{f_y}(w)}$$

The gain function can be interpreted as an OLS coefficient of a regression of y_t against the frequency component w of x_t. Gain is not a symmetric function. The coherence can be interpreted as the correlation between the frequency component w ofx_t with the frequency component w of y_t. Therefore, coherence is a symmetric function and it is also invariant to linear transformations.

The lag window imposes a greater weight for the autocovariance function located at lag 0 and decreasingly for higher lags or leads. Now, in case the cross autocovariance function does not have its peak at lag zero, then the estimate of the cross spectrum will be biased. Coherence will be especially biased due to this effect. The source of bias can be reduced by properly aligning the time series, such that after being aligned the cross-correlation function peaks at lag zero.

2 Monte Carlo simulation and the add-in

This section describes the data generator mechanism used to simulate a pair of time series and their population cross spectrum. In addition, the estimate of the cross spectrum of the data generator mechanism is presented. Consider the model:

$$y_t = \alpha x_{t-1} + e_t$$

$$x_t = \beta x_{t-1} + u_t$$

Where $e_t \sim nid(0,1)$, $u_t \sim nid(0,1)$ these processes are serially independent. In this way:

$$R_{xy}(k) = E(x_t y_{t+k}) = E(x_t(\alpha x_{t-1+k} + e_{t+k})) = \alpha R_x(k-1)$$

Therefore, the cross spectrum is determined by R_x as:

$$f_{xy}(w) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} R_{xy}(k)e^{-iwk} = \frac{\alpha}{2\pi}\sum_{k=-\infty}^{\infty} R_x(k-1)e^{-iwk}$$

If $j = k-1$ is defined

$$= \frac{\alpha}{2\pi}\sum_{j=-\infty}^{\infty} R_x(j)e^{-iw(j+1)} = \frac{\alpha e^{-iw}}{2\pi}\sum_{j=-\infty}^{\infty} R_x(j)e^{-iwj} = \alpha e^{-iw} f_x(w) = \alpha(cos(w) - isin(w))f_x(w)$$

In this way, the functions that describe the cross spectrum are determined by:

$$c_{xy}(w) = \alpha cos(w) f_x(w)$$

$$q_{xy}(w) = \alpha sin(w) f_x(w)$$

$$A_{xy}(w) = (c_{xy}(w)^2 + q_{xy}(w)^2)^{1/2} = |\alpha| f_x(w)(cos(w)^2 + sin(w)^2) = |\alpha| f_x(w)$$

$$\phi_{xy}(w) = tan^{-1}(\frac{-q_{xy}(w)}{c_{xy}(w)}) = tan^{-1}(-\frac{Sin(w)}{Cos(w)}) = tan^{-1}(-tan(w)) = -w$$

$$\frac{\phi_{xy}(w)}{w} = -1$$

$$G_{xy}(w) = \frac{A_{xy}(w)}{f_x(w)} = |\alpha|$$

Now considering the definition of y_t we have to:

$$f_y(w) = \alpha^2 f_x(w) + f_e(w)$$

$$G_{yx}(w) = \frac{|\alpha| f_x(w)}{\alpha^2 f_x(w) + f_e(w)}$$

$$K^2_{xy}(w) = \frac{|f_{xy}(w)|}{f_x(w) f_y(w)} = \frac{|\alpha|^2 f_x(w)}{f_x(w)(\alpha^2 f_x(w) + f_e(w))} = \frac{\alpha^2 f_x(w)}{\alpha^2 f_x(w)^2 + f_x(w) f_e(w)} = (1 + (f_e(w))/(\alpha^2 f_x(w)))^{-1}$$

The time series x_t is an AR(1) process, so $f_x(w)$ is given by:

$$f_x(w) = \frac{\sigma_u^2}{2\pi} |\frac{\beta(e^{iw})}{\alpha(e^{iw})}| = \frac{\sigma_u^2}{2\pi} (\frac{1}{1 + \beta^2 - 2\beta cos(w)})$$

$$f_e(w) = \frac{\sigma_e^2}{2\pi}$$

The following graphs present the population cross spectral functions described above and the estimates using the add-in. These are calculated with a value of $\alpha = \beta = 0.5$.

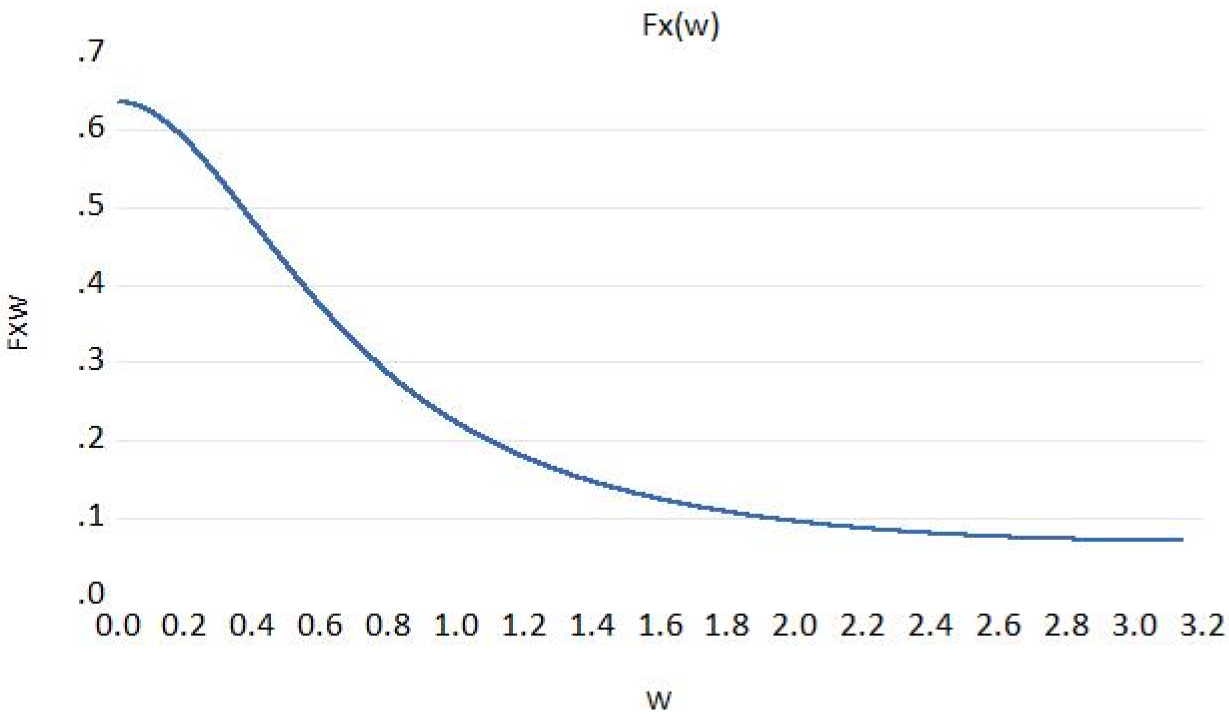

Figure 1: x_t spectrum $f_x(w)$

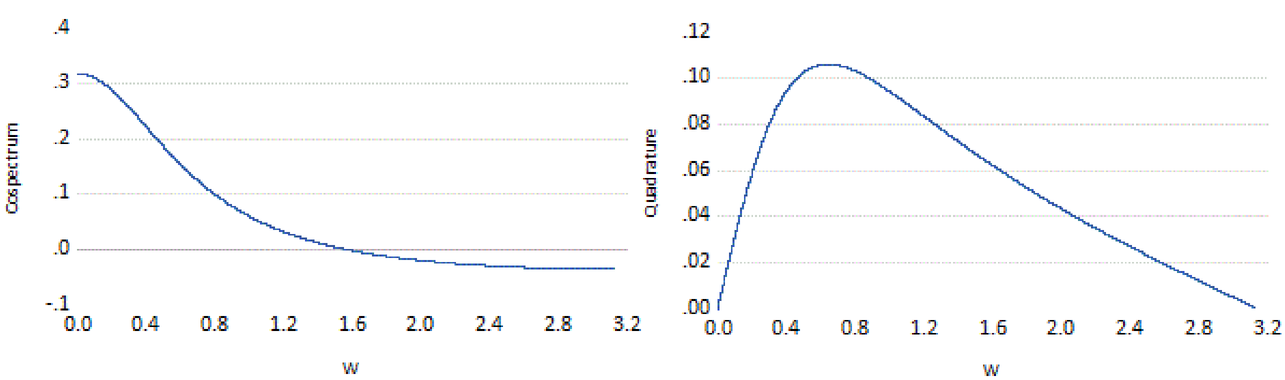

Figure 2: Cospectrum and quadrature spectrum

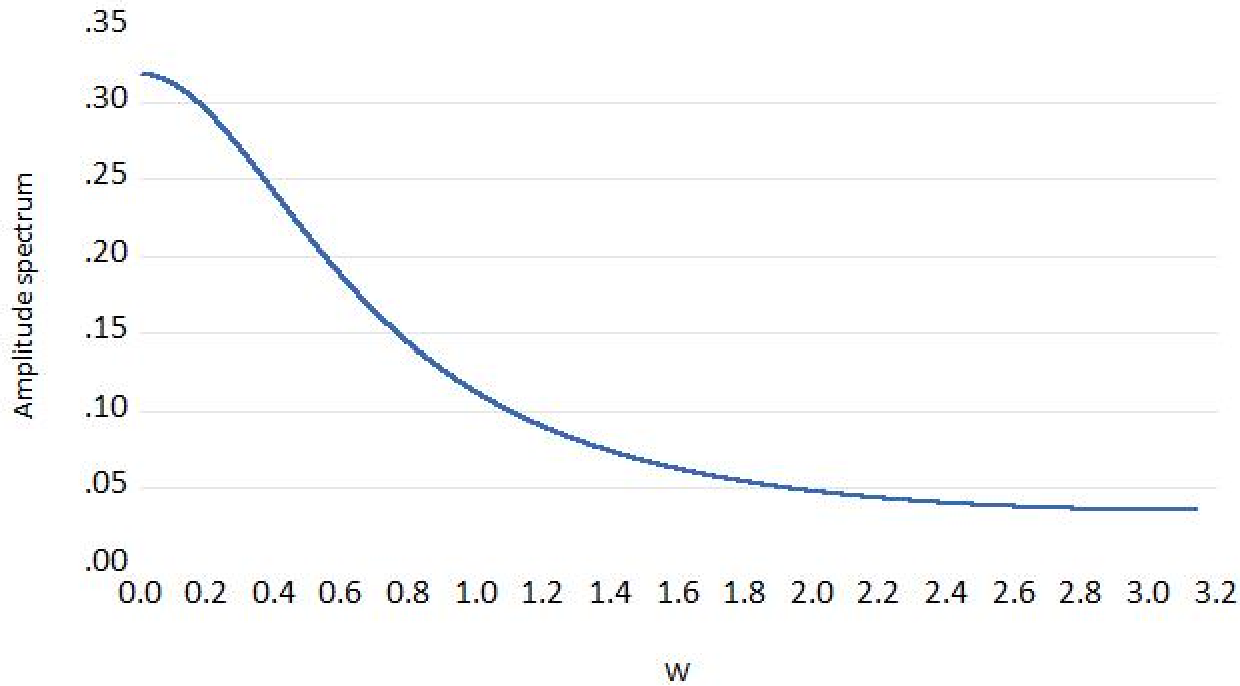

Figure 3: Amplitude spectrum

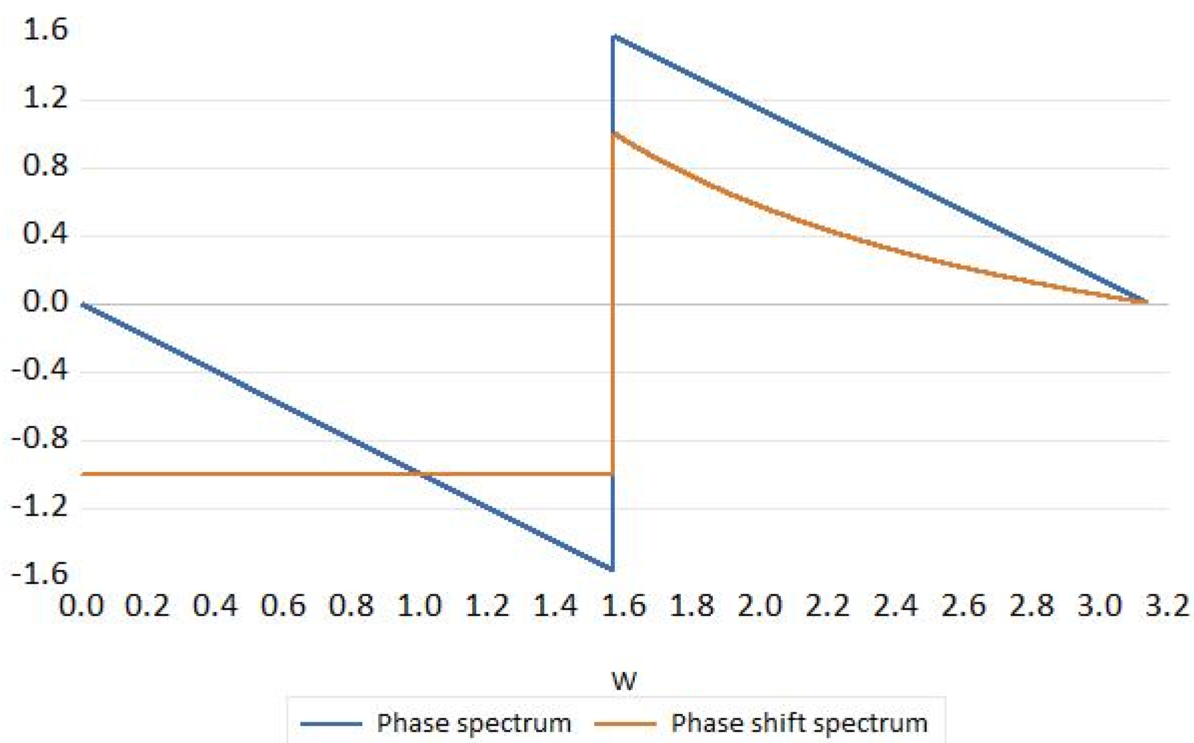

Figure 4: Phase and phase shift spectrum

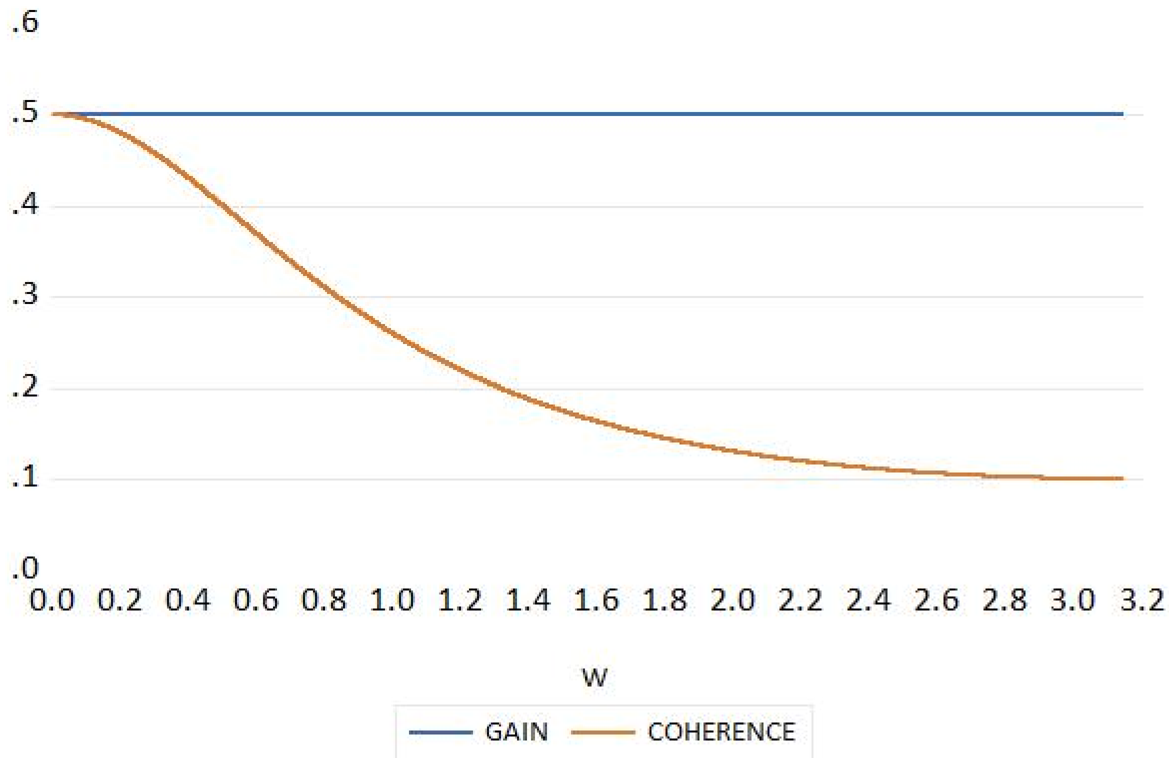

Figure 5: Gain and coherence

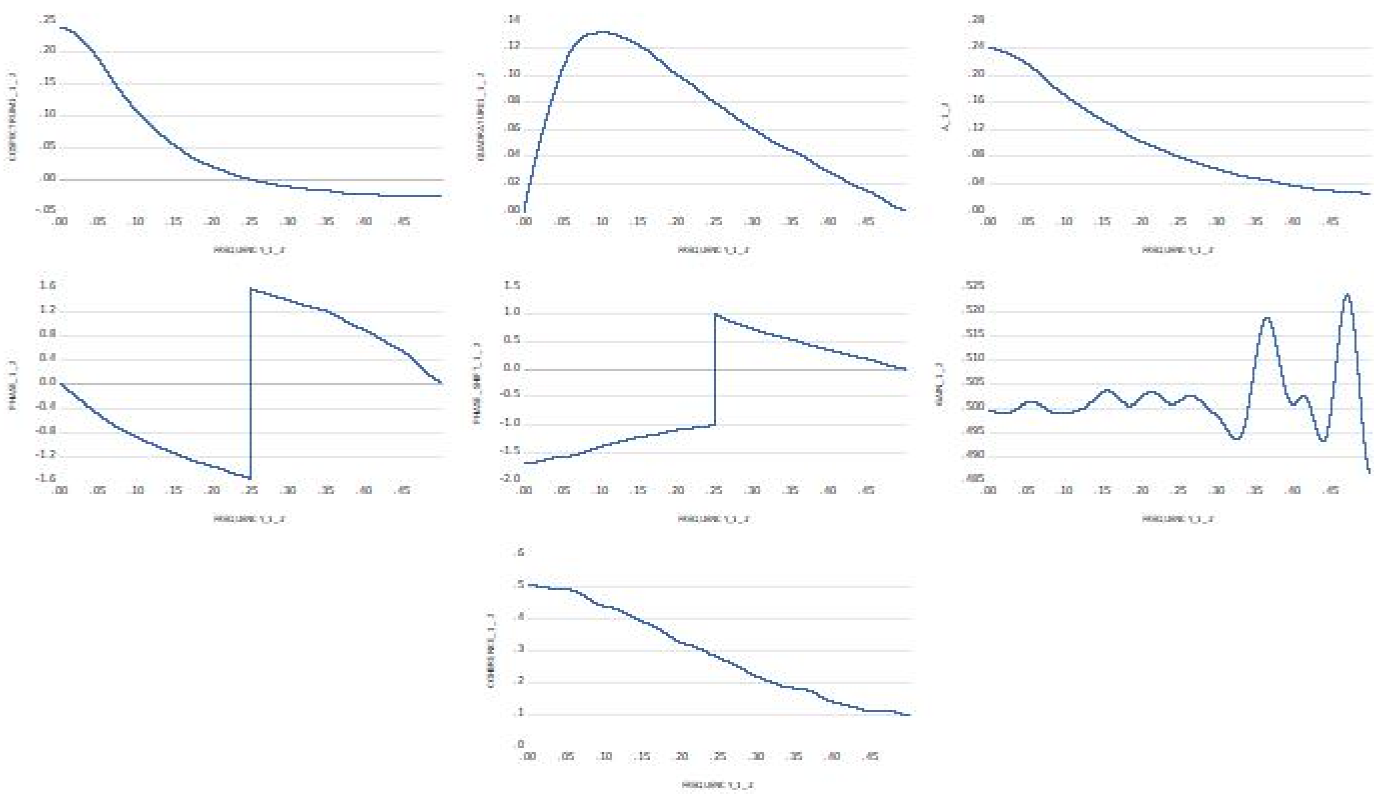

Figure 6: Estimated cross spectral functions using the add-in. Rectangular lag window. Truncation 20.

The following table shows the programming codes to use the add-in. The add-in must be used on a group of time series. For example the results in figure 6 can be obtained with the command group01.cspec(all). The add-in file contains the simulated data of the data generating process described above.

group_name.cspec(options)

Option	Command
Window	*Rectangular (default)*
	Hamming
	Hann
	Bartlett
	Parzen
Truncation	*truncation=number (20 default)*
Output table	*table*
Coherence	*Estimate coherence (default)*
Gain	*Estimate Gain (default)*
Phase_Shift	*Estimate Phase Shift (default)*
Phase	*Estimate Phase*
Amplitude	*Estimate Amplitude*
Cospectrum	*Estimate Cospectrum*
Quadrature	*Estimate Quadrature*
All	*Estimate all functions*
BC	*Perform bias correction (recommended for coherence bias)*

Table 2: Options for command line.

References

[1] Stoica, P., & Moses, R. L. (2005). Spectral analysis of signals.

[2] Priestley, M. B. (1981). Spectral analysis and time series: probability and mathematical statistics (No. 04; QA280, P7.).

[3] Wei, W. W. (2006). Time series analysis. In The Oxford Handbook of Quantitative Methods in Psychology: Vol. 2.