Fourth Moment Structure of the GARCH(p, q) Process

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First draft: January 1997

This version: February 1997

Working Paper Series in Economics and Finance No 168

Abstract. In this paper, a necessary and sufficient condition for the existence of the unconditional fourth moment of the GARCH(p, q) process is given as well as an expression for the moment itself. Furthermore, the autocorrelation function of the centred and squared observations of this process is derived. The statistical theory is further illustrated by a few special cases such as the GARCH(2,2) process and the ARCH(q) process.

Key Words. Autoregressive conditional heteroskedasticity, conditional variance, fat-tailed error distribution, time series, volatility

JEL Classification Code: C22

Acknowledgements. Financial support from the Bank of Sweden Tercentenary Foundation and the Swedish Council for Research in the Humanities and Social Sciences is gratefully acknowledged.

1 Introduction

The General Autoregressive Conditional Heteroskedasticity (GARCH) model and its many variants are popular in modelling volatility in high-frequency financial series. This is evident from the large number of recent surveys on ARCH and GARCH models and other ways of modelling volatility; see, for example, Bera and Higgins (1993), Bollerslev, Engle and Nelson (1994), Diebold and Lopez (1995), Engle (1995), Guégan (1994, ch. 5), Palm (1996) and Shephard (1996). The statistical properties of the basic GARCH model of Bollerslev (1986) have been discussed in a number of articles. Bollerslev (1986) derived conditions for the existence of unconditional moments of the GARCH(1,1) model under normality. He also derived expressions for these moments as functions of moments of lower order. Teräsvirta (1996) did the same without the normality assumption. Furthermore Bollerslev (1988) found the autocorrelation function for the GARCH(1,1) process with normal errors, and Teräsvirta (1996) generalized the result to symmetric non-normal error distributions. As for higher-order processes, Bollerslev (1986,1988), using Yule-Walker type equations, indicated the shape of the autocorrelation and partial autocorrelation functions of the general GARCH(p, q) process. Milhϕj (1985) already derived the autocorrelation function of an ARCH(q) process which is a special case ($p = 0$) of the GARCH(p, q) model.

Results on the existence of unconditional moments for GARCH models are not only of statistical interest. Practitioners may want to use them to see what kind of moment implications GARCH models they estimate may have. In particular, the existence of the unconditional fourth moment of stochastic processes generating, say, financial return data has interested researchers. Given both the existence conditions and a suitable expression for the fourth moment, the investigator would be able to check what his estimated model implies about this moment. Obtaining estimates of the kurtosis and the autocorrelations of the centred and squared observations from the model would also be useful. That would enable one to see how well the kurtosis and autocorrelation implied by the estimated model match the estimates obtained directly from the data. Such a comparison would give an indication of how well the model fits the data, because high kurtosis and slowly decaying autocorrelations of the centred and squared observations (or errors) are characteristic features of many financial high-frequency series. Note that the constancy of unconditional moments is a maintained assumption in this paper; for tests against weak stationarity, see Loretan and Phillips (1994).

The results of Bollerslev (1986,1988) and Teräsvirta (1996) make model evaluations of that type possible for the GARCH(1,1) model. This paper generalizes some of those results to GARCH(p, q) processes. The focus will be on the fourth moments and the autocorrelation function for the centred and squared observations.

The plan of the paper is as follows. In Section 2 we derive the necessary and sufficient condition for the existence of the unconditional fourth moment of the GARCH(p, p) process and give an expression for the moment itself. Section 3 presents the autocorrelation function for the squared process. In those sections, the GARCH(2,2) model is used as an illustration. Section 4 shows how these results are modified for the GARCH(p, q) model with $p \neq q$. Section 5 concludes.

2 Condition for existence of the fourth moment

We begin with a necessary and sufficient condition for existence of the unconditional fourth moment in the GARCH(p, p) model. The model is defined as

$$u_t = \varepsilon_t \sqrt{h_t} \tag{1}$$

where $\{\varepsilon_t\}$ is a sequence of independent identically distributed random variables with zero mean and a symmetric density. Furthermore,

$$h_t = \alpha_0 + \sum_{i=1}^{p} \beta_i h_{t-i} + \sum_{i=1}^{p} \alpha_i u_{t-i}^2 \tag{2}$$

see Bollerslev (1986). Assume now that this GARCH model has a finite fourth moment and let $\mathsf{E}\varepsilon_t^j = \nu_j$, $j = 2, 4$. Furthermore, let $\alpha_0 > 0$, $\alpha_i \geq 0$ and $\beta_i \geq 0$ for $i = 1, ..., p$, in (2). In particular, $\alpha_p > 0$ and $\beta_p > 0$. Let $c_{i,t-i} = \beta_i + \alpha_i \varepsilon_{t-i}^2$ for $i = 1, 2, ..., p$, where $\{c_{it}\}$ is a sequence of i.i.d. random variables such that c_{it} is independent of h_t. This allows us to rewrite (2) as

$$h_t = \alpha_0 + \sum_{i=1}^{p} c_{i,t-i} h_{t-i}. \tag{3}$$

In this paper we will make heavy use of (3). From the assumption that the second moments of $\{u_t\}$ exist it follows that $\sum_{i=1}^{p} \gamma_{i1} < 1$ where $\gamma_{i1} = \mathsf{E}c_{it} = \beta_i + \alpha_i \nu_2$, $i = 1, ..., p$. From (1) we see that in order to consider the unconditional fourth moment of $\{u_t\}$ we have to square (3), which gives

$$h_t^2 = \alpha_0^2 + 2\alpha_0 \sum_i c_{i,t-i} h_{t-i} + \sum_i c_{i,t-i}^2 h_{t-i}^2 + 2 \sum_{l<m} c_{l,t-l} c_{m,t-m} h_{t-l} h_{t-m}. \tag{4}$$

Taking the unconditional expectations on both sides of (4) yields

$$\mathsf{E}h_t^2 = \alpha_0^2 + 2\alpha_0 \gamma_1 \mathsf{E}h_t + \gamma_2 \mathsf{E}h_t^2 + 2 \sum_{l<m} \mathsf{E}(c_{l,t-l} c_{m,t-m} h_{t-l} h_{t-m}) \tag{5}$$

where $\mathsf{E}h_t = \alpha_0/(1-\gamma_1)$, $\gamma_i = \sum_{j=1}^{p} \gamma_{ji}$, $i = 1, 2$, and $\gamma_{i2} = \mathsf{E}c_{it}^2 = \beta_i^2 + 2\alpha_i \beta_i \nu_2 + \alpha_i^2 \nu_4$ for $i = 1, ..., p$. Thus, $\mathsf{E}h_t^2$ can be determined through (5) if we find a convenient expression for the mixed moments $\mathsf{E}(c_{l,t-l} c_{m,t-m} h_{t-l} h_{t-m})$. We shall tackle this problem by first finding a suitable expression for $h_t h_{t-n}$ and then using it in (5). This will be done in the next two subsections.

2.1 A useful representation of $h_t h_{t-n}$

First we introduce some notation for the case $1 \leq n \leq p$. Let $\mathbf{e}_r = (1, ..., 1, 0, ..., 0)^{'}$ be a $p \times 1$ vector with the first r components equal to 1. Furthermore, $\mathbf{c}_P = (c_{1,t-1}, ..., c_{p,t-p})^{'}$ is a $p \times 1$ vector with the index set $P = \{1, ..., p\}$, whereas $\mathbf{c}_{P \backslash \{n\}} = (c_{1,t-1}, ..., c_{n-1,t-n+1}, c_{n+1,t-n-1}, ..., c_{p,t-p})^{'}$ is a $(p-1) \times 1$ subvector of $\mathbf{c}_P$ obtained by excluding the element $c_{n,t-n}$ from $\mathbf{c}_P$. Matrix $\mathbf{I}_n$ is an $n \times n$ identity matrix, $\mathbf{0}_{m \times n}$ is an $m \times n$ null matrix and $\mathbf{0}_m$ is an $m \times 1$ null vector. We also define $(p-1) \times (p-1)$ matrices $\mathbf{C}_1 = \mathbf{I}_{p-1}$ and

$$\mathbf{C}_j = \begin{pmatrix} \mathbf{c}^{'}_{(P-1)\backslash\{n-j+1\}} & c_{p,t-j-p+1} \\ \mathbf{I}_{p-2} & \mathbf{0}_{p-2} \end{pmatrix} \tag{6}$$

for $j = 2, 3, ..., n$. Let $\mathbf{G}_{p-j}[c_{j+1,t-(i+j)+1}]$ be a $(p-j) \times (p-j)$ matrix generated by the "seed" element $c_{j+1,t-(i+j)+1}$. It has the form

$$\mathbf{G}_{p-j}[c_{j+1,t-(i+j)+1}] = \begin{pmatrix} \mathbf{c}^{'}_{(P-1)\backslash\{1,\ldots,j\}} & c_{p,t-i-p+2} \\ c_{j,t-i-j+2}\mathbf{I}_{p-j-1} & \mathbf{0}_{p-j-1} \end{pmatrix} \tag{7}$$

where $\mathbf{c}_{(P-1)\backslash\{1,\ldots,j\}} = (c_{j+1,t-(i+j-1)}, c_{j+2,t-(i+j)}, ..., c_{p-1,t-(i+p-3)})^{'}$ is a $(p-j-1) \times 1$ vector obtained by excluding the first j elements from $\mathbf{c}_{P-1}$. Let $p^* = p(p-1)/2$. Using (7), we define a $(p-1) \times p^*$ matrix $\mathbf{C}_{n+1}$ as

$$\mathbf{C}_{n+1} = \begin{pmatrix} \mathbf{G}_{p-1}[c_{2,t-n-2}] & \vdots & \mathbf{0}^{'}_{p-2} & \vdots \cdots \vdots & \mathbf{0}_{(p-3)\times 2} & \vdots & \mathbf{0}_{p-2} \\ & \vdots & \mathbf{G}_{p-2}[c_{3,t-n-3}] & \vdots \cdots \vdots & \mathbf{G}_2[c_{p-1,t-n-p+1}] & \vdots & \mathbf{G}_1[c_{p,t-n-p}] \end{pmatrix}. \tag{8}$$

Note that in (8) the number of rows of zero elements in each block of columns grows from zero to $p-2$ while the number of columns decreases from $p-1$ to one. Moreover, for $i > n+1$,

$$
\begin{aligned}
&\mathbf{C}_i \\
&= \begin{pmatrix}
\mathbf{G}_{p-1}[c_{2,t-i}] & \vdots & \mathbf{0}'_{p-2} & \vdots & \cdots & \vdots & \mathbf{0}_{(p-3)\times 2} & \vdots & \mathbf{0}_{p-2} \\
 & \vdots & \mathbf{G}_{p-2}[c_{3,t-i-1}] & \vdots & \cdots & \vdots & \mathbf{G}_2[c_{p-1,t-i-p+3}] & \vdots & \mathbf{G}_1[c_{p,t-i-p+1}] \\
\cdots\cdots\cdots & & \cdots\cdots\cdots & & \cdots & & \cdots\cdots\cdots & & \cdots\cdots\cdots \\
\mathbf{I}_{p-2}\vdots\mathbf{0}_{p-2} & \vdots & \mathbf{0}_{(p-2)\times(p-2)} & \vdots & \cdots & \vdots & 0 & 0 & 0 \\
\mathbf{0}_{(p-3)\times(p-1)} & \vdots & \mathbf{I}_{p-3}\vdots\mathbf{0}_{p-3} & \vdots & & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & 0 & \vdots & \vdots \\
\mathbf{0}'_{p-1} & \vdots & \mathbf{0}'_{p-2} & \vdots & & \vdots & 1 & 0 & 0
\end{pmatrix}
\end{aligned}
\tag{9}
$$

which is a $p^* \times p^*$ matrix.

When $n > p$, we have to redefine some of the notation as follows. Let $\mathbf{c}^*_{p+1} = (\alpha_0, c_{1,t-1}, ..., c_{p,t-p})'$ be a $(p+1) \times 1$ vector. We define the $(p+1) \times (p+1)$ matrices $\mathbf{C}^*_1 = \mathbf{I}_{p+1}$ and

$$
\mathbf{C}^*_j = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 \\
\alpha_0 & c_{1,t-j} & c_{2,t-j-1} & \cdots & c_{p-1,t-j-p+2} & c_{p,t-j-p+1} \\
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0
\end{pmatrix}
\tag{10}
$$

for $j = 2, ..., n$. Furthermore, using (7), we define the $(p+1) \times p^*$ matrix $\mathbf{C}^*_{n+1}$ as

$$
\begin{aligned}
&\mathbf{C}^*_{n+1} \\
&= \begin{pmatrix}
\mathbf{0}_{2\times(p-1)} & \vdots & \mathbf{0}_{3\times(p-2)} & \vdots\cdots\vdots & \mathbf{0}_{(p-1)\times 2} & \vdots & \mathbf{0}_{p\times 1} \\
\mathbf{G}_{p-1}\left[c_{2,t-n-2}\right] & \vdots & \mathbf{G}_{p-2}\left[c_{3,t-n-3}\right] & \vdots\cdots\vdots & \mathbf{G}_2\left[c_{p-1,t-n-p+1}\right] & \vdots & \mathbf{G}_1\left[c_{p,t-n-p}\right]
\end{pmatrix}.
\end{aligned}
\tag{11}
$$

Finally, $\mathbf{C}^*_i = \mathbf{C}_i$ for $i > n+1$.

Now we are ready to consider $h_t h_{t-n}$, $n \geq 1$. We start a recursion by applying (3) to

h_t. This yields an expression which is dependent on h_{t-1} but no longer on h_t. Applying (3) to h_{t-1} completes the next recursion. After the kth recursion, $k \geq n+1$, we have

Lemma 1. *Let $n \geq 1$. For $k \geq n+1$, $h_t h_{t-n}$ can be expressed by combinations of the terms of h_{t-i}, h_{t-j}^2 and $h_{t-i}h_{t-j}$ such that*

$$h_t h_{t-n} = S_{10} + \sum_{i=n+1}^{k} S_{1i} + S_{20} + \sum_{i=n+1}^{k} S_{2i} + S_k \tag{12}$$

where, for $1 \leq n \leq p$,

$$\begin{aligned}
S_{10} &= \alpha_0 \left[\left(1 + \mathbf{c}'_{P\backslash\{n\}} \sum_{i=1}^{n-1} \left(\prod_{j=1}^{i} \mathbf{C}_j \right) \mathbf{e}_1 \right) h_{t-n} + \mathbf{c}'_{P\backslash\{n\}} \left(\prod_{i=1}^{n} \mathbf{C}_i \right) \mathbf{h}_{10t} \right] \\
S_{1i} &= \alpha_0 \mathbf{c}'_{P\backslash\{n\}} \left(\prod_{j=1}^{i} \mathbf{C}_j \right) \mathbf{h}_{1it} \\
S_{20} &= \left(c_{n,t-n} + \mathbf{c}'_{P\backslash\{n\}} \sum_{i=1}^{n-1} \left(\prod_{j=1}^{i} \mathbf{C}_j \right) \mathbf{e}_1 c_{n-i,t-n} \right) h_{t-n}^2 \\
&\quad + \mathbf{c}'_{P\backslash\{n\}} \left(\prod_{i=1}^{n} \mathbf{C}_i \right) \mathbf{h}_{20t} \\
S_{2i} &= \mathbf{c}'_{P\backslash\{n\}} \left(\prod_{j=1}^{i} \mathbf{C}_j \right) \mathbf{h}_{2it} \\
S_k &= \mathbf{c}'_{P\backslash\{n\}} \left(\prod_{j=1}^{k+1} \mathbf{C}_j \right) \mathbf{h}_{kt}
\end{aligned} \tag{13}$$

and for $n > p$,

$$\begin{aligned}
S_{10} &= \mathbf{c}^{*\prime}_{p+1} \left(\prod_{i=1}^{n} \mathbf{C}^*_i \right) \mathbf{h}^*_{10t} \\
S_{1i} &= \alpha_0 \mathbf{c}^{*\prime}_{p+1} \left(\prod_{j=1}^{i} \mathbf{C}^*_j \right) \mathbf{h}_{1it} \\
S_{20} &= \mathbf{c}^{*\prime}_{p+1} \left(\prod_{i=1}^{n} \mathbf{C}^*_i \right) \mathbf{h}^*_{20t} \\
S_{2i} &= \mathbf{c}^{*\prime}_{p+1} \left(\prod_{j=1}^{i} \mathbf{C}^*_j \right) \mathbf{h}_{2it} \\
S_k &= \mathbf{c}^{*\prime}_{p+1} \left(\prod_{j=1}^{k+1} \mathbf{C}^*_j \right) \mathbf{h}_{kt}.
\end{aligned} \tag{14}$$

In (13), $\mathbf{h}_{10t}$ *and* $\mathbf{h}_{20t}$ *are* $(p-1)\times 1$ *vectors given by*

$$\begin{aligned}\mathbf{h}_{10t} &= \left(h_{t-(n+1)},...,h_{t-(n+p-1)}\right)' \\ \mathbf{h}_{20t} &= \left(c_{1,t-(n+1)}h^2_{t-(n+1)},...,c_{p-1,t-(n+p-1)}h^2_{t-(n+p-1)}\right)'\end{aligned} \tag{15}$$

whereas in (14), $\mathbf{h}^*_{10t}$ *and* $\mathbf{h}^*_{20t}$ *are* $(p+1)\times 1$ *vectors given by*

$$\begin{aligned}\mathbf{h}^*_{10t} &= \left(h_{t-n},0,\alpha_0 h_{t-(n+1)},...,\alpha_0 h_{t-(n+p-1)}\right)' \\ \mathbf{h}^*_{20t} &= \left(0,h^2_{t-n},c_{1,t-(n+1)}h^2_{t-(n+1)},...,c_{p-1,t-(n+p-1)}h^2_{t-(n+p-1)}\right)'.\end{aligned} \tag{16}$$

Finally, $\mathbf{h}_{1it}$ *,* $\mathbf{h}_{2it}$ *and* $\mathbf{h}_{kt}$ *are* $p^*\times 1$ *vectors given by*

$$\begin{aligned}\mathbf{h}_{1it} &= \left(h_{t-i},...,h_{t-(i+p-2)},0,...,0\right)' \\ \mathbf{h}_{2it} &= \left(c_{1,t-i}h^2_{t-i},...,c_{p-1,t-(i+p-2)}h^2_{t-(i+p-2)},0,...,0\right)' \\ \mathbf{h}_{kt} &= (h_{t-k}h_{t-(k+1)},\cdots,h_{t-k}h_{t-(k+p-1)},h_{t-(k+1)}h_{t-(k+2)}, \\ &\quad \cdots,h_{t-(k+1)}h_{t-(k+p-1)},\cdots,h_{t-(k+p-2)}h_{t-(k+p-1)})'.\end{aligned} \tag{17}$$

Proof. See Appendix 1.

Continuing the recursion it follows from Lemma 1 that $h_t h_{t-n}$, $1\le n\le p$, is characterized by matrices $\mathbf{C}_1$, $\mathbf{C}_2$, ..., $\mathbf{C}_n$, $\mathbf{C}_{n+1}$ and, finally, $\mathbf{C}_k$ for $k>n+1$. S_{10} and S_{20} are initial terms that are only functions of $\mathbf{C}_1$, $\mathbf{C}_2$, ..., $\mathbf{C}_n$. S_{1i}, S_{2i} and S_k also depend on $\mathbf{C}_k$ when $k\ge n+1$. Moreover, $h_t h_{t-n}$, $n>p$, is characterized by matrices $\mathbf{C}^*_1$, $\mathbf{C}^*_2$, ..., $\mathbf{C}^*_n$, $\mathbf{C}^*_{n+1}$ and $\mathbf{C}^*_k$ for $k>n+1$. These results are what we need for handling the mixed moments in (5).

2.2 The mixed moment $\mathsf{E}(c_{l,t-l}c_{m,t-m}h_{t-l}h_{t-m})$

Let $1\le l<m\le p$ and consider $\mathsf{E}(c_{l,t-l}c_{t-m}h_{t-l}h_{t-m})$, $l<m$. Substituting $t-l$ for t and $m-l$ for n in Lemma 1, we obtain a recursion formula for $h_{t-l}h_{t-m}$. Let $\mathsf{E}\mathbf{c}_P=\boldsymbol{\gamma}_P=$

$(\gamma_{11}, ..., \gamma_{p1})'$, $\mathsf{E}\mathbf{c}_{P\backslash\{m-l\}} = \boldsymbol{\gamma}_{P\backslash\{m-l\}} = (\gamma_{11}, ..., \gamma_{m-l-1}, \gamma_{m-l+1}, ..., \gamma_{p1})'$, $\mathsf{E}\mathbf{C}_k = \mathbf{\Gamma}_k$ for $k = 1, 2, ..., m-l+1$ and $\mathsf{E}\mathbf{C}_k = \mathbf{\Gamma}$ for $k > m-l+1$. Thus, $\mathbf{\Gamma}_k, k = 1, 2, ..., m-l$, are matrices of order $(p-1) \times (p-1)$ and, in particular, $\mathbf{\Gamma}_1 = \mathbf{I}_{p-1}$. Furthermore, $\mathbf{\Gamma}_{m-l+1}$ is of order $(p-1) \times p^*$, and $\mathbf{\Gamma}$ is a $p^* \times p^*$ matrix. The elements of those matrices are functions of $\gamma_{i1} = \mathsf{E}c_{it}$, $i = 1, ..., p$. In addition, let

$$\lambda(\mathbf{\Gamma}) = \max\{|\lambda_i|\} \tag{18}$$

be the maximum absolute eigenvalue of the matrix $\mathbf{\Gamma}$ where λ_i, $i = 1, ..., p^*$, are the eigenvalues of matrix $\mathbf{\Gamma}$.

Applying Lemma 1 we have that

$$\begin{aligned} & \mathsf{E}(c_{l,t-l}c_{t-m}h_{t-l}h_{t-m}) \\ = \ & \mathsf{E}(c_{l,t-l}c_{m,t-m}S_{10}) + \mathsf{E}(c_{l,t-l}c_{m,t-m}S_{20}) + \sum_{i=m-l+1}^{k} \mathsf{E}(c_{l,t-l}c_{m,t-m}S_{1i}) \\ & + \sum_{i=m-l+1}^{k} \mathsf{E}(c_{l,t-l}c_{m,t-m}S_{2i}) + \mathsf{E}(c_{l,t-l}c_{m,t-m}S_k). \end{aligned} \tag{19}$$

Assume that the process started at some finite value infinitely many periods ago. It turns out that the limit of (19) exists and is independent of t as $k \to \infty$ if and only if all the eigenvalues of $\mathbf{\Gamma}$ lie inside the unit circle, that is, $\lambda(\mathbf{\Gamma})<1$. We show this and derive the mixed moment $\mathsf{E}(c_{l,t-l}c_{t-m}h_{t-l}h_{t-m})$ as a function of $\mathsf{E}h_t$ and $\mathsf{E}h_t^2$ by proving the next three lemmata.

Lemma 2. *For* $1 \leq l < m \leq p$,

$$\lim_{k\to\infty} \mathsf{E}(c_{l,t-l}c_{m,t-m}S_k) = 0 \tag{20}$$

if and only if $\lambda(\mathbf{\Gamma}) < 1$.

Proof. See Appendix 2 .

Lemma 3. *For* $1 \leq l < m \leq p$,

$$\mathsf{E}\left(c_{l,t-l}c_{m,t-m}\sum_{i=m-l+1}^{\infty}S_{1i}\right) = \alpha_0\gamma_{l1}\gamma_{m1}\boldsymbol{\gamma}'_{P\backslash\{m-l\}}\left(\prod_{i=1}^{m-l+1}\boldsymbol{\Gamma}_i\right)\left(\mathbf{I}_{p^*}-\boldsymbol{\Gamma}\right)^{-1}\mathbf{e}_{p-1}\mathsf{E}h_t \tag{21}$$

if and only if $\lambda(\boldsymbol{\Gamma}) < 1$*, where* $\mathbf{e}_{p-1} = (1,\cdots,1,0,\cdots,0)'$ *is a* $p^* \times 1$ *vector with the first* $p-1$ *elements equal to* 1.

Proof. See Appendix 3 .

Evaluating $\mathsf{E}\left(c_{l,t-l}c_{m,t-m}\sum_{i=m-l+1}^{\infty}S_{2i}\right)$ is not an easy task because $\mathbf{h}_{2it}$ is not stochastically independent of the matrix product $\mathbf{C}_{i-(p-2)}\cdots\mathbf{C}_i$. However, for each nonzero element of $\mathbf{h}_{2it}$, $c_{j,t-(i+j)}$ and $h^2_{t-(i+j)}$ are stochastically independent. For $j \leq p-2$ and $i > m-l+2p-1$, this allows us to consider

$$\boldsymbol{\gamma}(i-j,i) = \mathsf{E}(\mathbf{C}_{i-j}\mathbf{C}_{i-(j+1)}\cdots\mathbf{C}_i\mathbf{c}_{2it}) \tag{22}$$

where $\mathbf{c}_{2it} = (c_{1,t-(i+l)},...,c_{p-1,t-(i+l+p-2)},0,...,0)'$ is a $p^* \times 1$ vector whose first $p-1$ elements are nonzero. We see that $\boldsymbol{\gamma}(i-j,i)$ is a $p^* \times 1$ column vector which is determined by the matrix product $\mathbf{C}_{i-j}\cdots\mathbf{C}_i$ and whose elements consist of $\widetilde{\gamma}_{ij} = \mathsf{E}(c_{it}c_{jt})$ for $i < j$. Furthermore, for $j = l < i \leq p$, define

$$\gamma(\mathbf{c}(m-l),i-l-1,i) = \mathsf{E}(\mathbf{c}'_{P\backslash\{m-l\}}\mathbf{C}_{i-(l+1)}\cdots\mathbf{C}_i\mathbf{c}_{2it}) \tag{23}$$

where $\mathbf{c}(x)$ is shorthand for $\mathbf{c}_{P\backslash\{x\}}$. Using (22) and (23), we have

Lemma 4. *For* $1 \leq l < m \leq p$,

$$\begin{aligned}
&\mathsf{E}\left(c_{l,t-l}c_{m,t-m}\sum_{i=m-l+1}^{\infty}S_{2i}\right) \\
= \;& \gamma_{l1}\gamma_{m1}\sum_{i=1}^{p-1}\gamma(\mathbf{c}(m-l),2,m-l+i-1)\mathsf{E}h_t^2 + \gamma_{l1}\gamma_{m1}\boldsymbol{\gamma}'_{P\backslash\{m-l\}}\left(\prod_{i=1}^{m-l+1}\boldsymbol{\Gamma}_i\right) \\
&\times\left(\mathbf{I}_{p^*}-\boldsymbol{\Gamma}\right)^{-1}\boldsymbol{\gamma}(m-l+p+1,m-l+2p-1)\mathsf{E}h_t^2
\end{aligned} \tag{24}$$

if and only if $\lambda(\boldsymbol{\Gamma}) < 1$.

Proof. See Appendix 4.

It now follows from these three lemmata that if the condition $\lambda(\mathbf{\Gamma}) < 1$ holds the mixed moment $\mathsf{E}(c_{l,t-l}c_{t-m}h_{t-l}h_{t-m})$ converges to a finite value which is a linear function of $\mathsf{E}h_t$ and $\mathsf{E}h_t^2$ as $k \to \infty$. Combining these results and using (19) yields

Theorem 1. *Assume that $\lambda(\mathbf{\Gamma}) < 1$. Under this condition,*

$$\mathsf{E}(c_{l,t-l}c_{m,t-m}h_{t-l}h_{t-m}) = \alpha_0\gamma_{l1}\gamma_{m1}M_1(l,m)\mathsf{E}h_t + \gamma_{l1}M_2(l,m)\mathsf{E}h_t^2 \tag{25}$$

where, for $m-l>1$,

$$M_1(l,m) = 1+\boldsymbol{\gamma}'_{P\backslash\{m-l\}}\left[\sum_{i=1}^{m-l-1}\left(\prod_{j=1}^{i}\mathbf{\Gamma}_j\right)\mathbf{e}_1+\prod_{i=1}^{m-l}\mathbf{\Gamma}_i\left(\mathbf{j}_{p-1}+\mathbf{\Gamma}_{m-l+1}\left(\mathbf{I}_{p^*}-\mathbf{\Gamma}\right)^{-1}\mathbf{e}_{p-1}\right)\right] \tag{26}$$

and, in particular,

$$M_1(m-1,m) = 1+\boldsymbol{\gamma}'_{P\backslash\{1\}}\left[\mathbf{j}_{p-1}+\mathbf{\Gamma}_2\left(\mathbf{I}_{p^*}-\mathbf{\Gamma}\right)^{-1}\mathbf{e}_{p-1}\right]$$

where $\mathbf{j}_{p-1} = (1,1,...,1)'$ is a $(p-1)\times 1$ vector. Furthermore,

$$M_2(l,m) = M_{21}(l,m) + \gamma_{m1}\sum_{i=2}^{4}M_{2i}(l,m) \tag{27}$$

such that for $m-l>1$

$$M_{21}(l,m) = \widetilde{\gamma}_{m-l,m} + \boldsymbol{\gamma}'_{P\backslash\{m-l\}}\left[\sum_{i=1}^{m-l-1}\left(\prod_{j=1}^{i}\mathbf{\Gamma}_j\right)\mathbf{e}_1\widetilde{\gamma}_{m-l-i,m}\right] \tag{28}$$

$$M_{22}(l,m) = \sum_{i=1}^{m-l-1}\gamma_{i1}M_{22}(m-l-i) + \sum_{j=m-l+1}^{p}\widetilde{\gamma}_{j-m+l,j} \tag{29}$$

and, for $m-l=1$, $M_{21}(m-1,m) = \widetilde{\gamma}_{1m}$ and $M_{22}(m-1,m) = \sum_{j=2}^{p}\widetilde{\gamma}_{j-1,j}$. In addition, in (27), for any $l<m$,

$$M_{23}(l,m) = \sum_{i=m-l+1}^{m-l+p-1}\gamma\left(\mathbf{c}(m-l),2,i-1\right) \tag{30}$$

$$M_{24}(l,m) = \boldsymbol{\gamma}'_{P\backslash\{m-l\}}\left(\prod_{j=1}^{m-l+1}\mathbf{\Gamma}_j\right)\left(\mathbf{I}_{p^*}-\mathbf{\Gamma}\right)^{-1}\boldsymbol{\gamma}\left(m-l+p+1,m-l+2p-1\right). \tag{31}$$

Proof. See Appendix 5.

2.3 A necessary and sufficient condition for existence of the fourth moment

Theorem 1 allows us to express the expectation (5) as a function of $\mathsf{E}h_t$ and $\mathsf{E}h_t^2$. We can write

$$\mathsf{E}h_t^2 = \alpha_0^2 + 2\alpha_0 \left(\gamma_1 + \sum_{l<m} \gamma_{l1}\gamma_{m1} M_1(l,m) \right) \mathsf{E}h_t + \left(\gamma_2 + 2\sum_{l<m} \gamma_{l1} M_2(l,m) \right) \mathsf{E}h_t^2. \quad (32)$$

From (32) and the assumption $\lambda(\mathbf{\Gamma}) < 1$ we have

Theorem 2. *A necessary and sufficient condition for the existence of the fourth unconditional moment of $\{u_t\}$ in the GARCH(p,p) model (1) with (2) is*

$$\gamma_2 + 2\sum_{l<m} \gamma_{l1} M_2(l,m) < 1. \quad (33)$$

The fourth moment is

$$\mathsf{E}(u_t^4) = \frac{\alpha_0^2 \nu_4 [1 + \gamma_1 + 2\sum_{l<m} \gamma_{l1}\gamma_{m1} M_1(l,m)]}{(1-\gamma_1)[1 - \gamma_2 - 2\sum_{l<m} \gamma_{l1} M_2(l,m)]}. \quad (34)$$

2.4 Special case: GARCH(2,2)

To illustrate the above general theory we consider the GARCH(2,2) process. We have

Corollary 1. *For the GARCH$(2,2)$ model,*

$$\mathsf{E}(c_{1,t-1}c_{2,t-2}h_{t-1}h_{t-2}) = \alpha_0\gamma_{11}\gamma_{21}M_1(1,2)\,\mathsf{E}h_t + \gamma_{11}M_2(1,2)\mathsf{E}h_t^2 \quad (35)$$

where $M_1(1,2) = (1-\gamma_{21})^{-1}$ *and* $M_2(1,2) = \widetilde{\gamma}_{12}(1-\gamma_{21})^{-1}$ if and only if $\gamma_{21} < 1$.

Proof. See Appendix 6.

Corollary 2. *For the GARCH$(2,2)$ model a necessary and sufficient condition for the existence of the unconditional fourth moment of $\{u_t\}$ is that*

$$(\gamma_{12} + \gamma_{22})(1-\gamma_{21}) + 2\gamma_{11}\widetilde{\gamma}_{12} + \gamma_{21} < 1. \quad (36)$$

When (36) holds,

$$\mathsf{E}(u_t^4) = \frac{\alpha_0^2 \nu_4 \left[(1+\gamma_{11}+\gamma_{21})(1-\gamma_{21})+2\gamma_{11}\gamma_{21}\right]}{(1-\gamma_{11}-\gamma_{21})\left[(1-\gamma_{21})(1-\gamma_{12}-\gamma_{22})-2\gamma_{11}\widetilde{\gamma}_{12}\right]}. \tag{37}$$

If $\beta_2 = \alpha_2 = 0$, we have the GARCH(1,1) model, and (36) is simply

$$\gamma_{12} = \beta_1^2 + 2\beta_1\alpha_1\nu_2 + \alpha_1^2\nu_4 < 1. \tag{38}$$

This is the existence condition in Teräsvirta (1996). Setting $\nu_2 = 1$ and $\nu_4 = 3$ in (38) yields the condition which Bollerslev (1986) obtained for normal errors.

3 The autocorrelation function for the squared process

Milhϕj (1985) proved that the squared process $\{u_t^2\}$ has an autocorrelation function similar to that of a standard autoregressive process of order p when $\{u_t\}$ follows an ARCH(p) process with normal errors. He also defined the first p autocorrelations of $\{u_t^2\}$ as

$$\boldsymbol{\rho} = (\mathbf{I}_p - \boldsymbol{\Psi})^{-1} \boldsymbol{\alpha} \tag{39}$$

where $\boldsymbol{\rho} = (\rho_1, ..., \rho_p)'$, $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_p)'$ and the $p \times p$ matrix $\boldsymbol{\Psi} = (\psi_{ij})$ is defined by $\psi_{ij} = \alpha_{i+j} + \alpha_{i-j}$ making use of the fact that $\alpha_k = 0$ for $k \leq 0$ and $k > p$. For $n > p$,

$$\rho_n = \sum_{i=1}^{p} \alpha_i \rho_{n-i}. \tag{40}$$

Bollerslev (1986,1988) applied this approach to the GARCH(p, p) case and obtained the analogue of the Yule-Walker equations for the autocorrelation function of $\{u_t^2\}$. He was able to show that for any $n > p$, the autocorrelations have the form

$$\rho_n = \sum_{i=1}^{p} (\beta_i + \alpha_i)\, \rho_{n-i}. \tag{41}$$

However, he did not yet provide the first p autocorrelations $\rho_1, ..., \rho_p$. We shall do that here.

To fix notation, write the nth order autocorrelation of $\{u_t^2\}$ as

$$\rho_n = \rho(u_t^2, u_{t-n}^2) = \frac{\mathsf{E}(u_t^2 u_{t-n}^2) - (\mathsf{E}u_t^2)^2}{\mathsf{E}(u_t^4) - (\mathsf{E}u_t^2)^2} \tag{42}$$

for $n \geq 1$. In order to obtain ρ_n, we must find an expression for $\mathsf{E}(u_t^2 u_{t-n}^2)$ as a function of $\mathsf{E}u_t^4$ and $\mathsf{E}u_t^2$. This can be done by applying results in Section 2.

3.1 The mixed moment $\mathsf{E}(u_t^2 u_{t-n}^2)$

It is obvious that combining (1) and Lemma 1 will give us the mixed moment $\mathsf{E}(u_t^2 u_{t-n}^2)$, $n \geq 1$. Consider first the case $1 \leq n \leq p$. Let $\overline{\gamma}_{i1} = \mathsf{E}(\varepsilon_t^2 c_{it})$ for $i = 1, ..., p$. Setting $M_1(n) = M_1(0, n)$ and $M_2(n) = M_2(0, n)$, it follows directly from Theorem 1 that

$$\mathsf{E}\left(u_t^2 u_{t-n}^2\right) = \alpha_0 \nu_2^2 M_1(n)\, \mathsf{E}h_t + \nu_2 M_2(n)\, \mathsf{E}h_t^2. \tag{43}$$

As in (27), let

$$M_2(n) = M_{21}(n) + \nu_2 \sum_{i=2}^{4} M_{2i}(n) \tag{44}$$

such that for $n = 2, ..., p$,

$$M_{21}(n) = \overline{\gamma}_{n1} + \boldsymbol{\gamma}_{P\backslash\{n\}}^{'} \left[\sum_{i=1}^{n-1} \left(\prod_{j=1}^{i} \boldsymbol{\Gamma}_j\right) \mathbf{e}_1 \overline{\gamma}_{n-i,1}\right] \tag{45}$$

$$M_{22}(n) = \sum_{i=1}^{n-1} \gamma_{i1} M_{22}(n-i) + \sum_{j=n+1}^{p} \widetilde{\gamma}_{j-n,j}. \tag{46}$$

Furthermore, for $n = 1$, $M_{21}(1) = \overline{\gamma}_{11}$ and $M_{22}(1) = \sum_{j=2}^{p} \widetilde{\gamma}_{j-1,j}$. In addition, for $n = 1, ..., p$,

$$M_{23}(n) = \sum_{i=n+1}^{n+p-1} \gamma(\mathbf{c}(n), 2, i-1) \tag{47}$$

and

$$M_{24}(n) = \boldsymbol{\gamma}_{P\backslash\{n\}}^{'} \left[\left(\prod_{j=1}^{n+1} \boldsymbol{\Gamma}_j\right) (\mathbf{I}_{p*} - \boldsymbol{\Gamma})^{-1}\right] \boldsymbol{\gamma}(n+p+1, n+2p-1). \tag{48}$$

Next, let $n > p$. Under this assumption it can be shown that (43) still holds if we define the coefficients of $\mathsf{E}h_t$ and $\mathsf{E}h_t^2$ as functions of n in a proper way. In order to do that, first note an analogy to the situation in Section 2. Thus, while $c_{i,t-(n+j)}$ and $h^2_{t-(n+j)}$ are stochastically independent, the product $\mathbf{C}^*_{n-(p-1)}\mathbf{C}^*_{n-(p-2)}\cdots\mathbf{C}^*_n$ is not stochastically independent of $\mathbf{h}^*_{20t}$. This is because there is an element such as $c_{i,t-n}$ which is a function of ε^2_{t-n} for some n on the second row of each matrix, say, $\mathbf{C}^*_{n-j}$, $j=0,1,...,p-1$. As in Section 2, define the $(p+1)\times 1$ vector $\mathbf{c}^*_{20t}=\left(0,1,c_{1,t-(n+1)},...,c_{p-1,t-(n+p-1)}\right)'$. We now have to consider the expectation $\mathsf{E}\left(\varepsilon^2_{t-n}\mathbf{C}^*_{n-(p-1)}\mathbf{C}^*_{n-(p-2)}\cdots\mathbf{C}^*_n\mathbf{c}^*_{20t}\right)$, $n>p$, which is given in the following lemma.

Lemma 5 . *For $n>p$,*

$$\begin{aligned}&\mathsf{E}\left(\varepsilon^2_{t-n}\mathbf{C}^*_{n-(p-1)}\mathbf{C}^*_{n-(p-2)}\cdots\mathbf{C}^*_n\mathbf{c}^*_{20t}\right)\\ =\;&(0,M_{21}(p)+\nu_2M_{22}(p),M_{21}(p-1)+\nu_2M_{22}(p-1),\cdots,M_{21}(1)+\nu_2M_{22}(1))'. \quad (49)\end{aligned}$$

Proof. See Appendix 7.

Lemma 5 implies that for $n>p$, $\mathsf{E}\left(\varepsilon^2_{t-n}\mathbf{C}^*_{n-(p-1)}\mathbf{C}^*_{n-(p-2)}\cdots\mathbf{C}^*_n\mathbf{h}^*_{20t}\right)$ does not depend on n.

Let $\mathsf{E}\mathbf{c}^*_{p+1}=\boldsymbol{\gamma}_{p+1}=\left(\alpha_0,\gamma_{11},...,\gamma_{p1}\right)'$ be a $(p+1)\times 1$ vector, $\mathsf{E}\mathbf{C}^*_i=\mathbf{\Gamma}_*$ for $i=2,...,n$, $\mathsf{E}\mathbf{C}^*_{n+1}=\mathbf{\Gamma}^*_{n+1}$ and $\mathsf{E}\mathbf{C}^*_i=\mathbf{\Gamma}$ for $i>n+1$. We observe that here $\mathsf{E}\mathbf{C}^*_i$ $(i=2,...,n+1)$ are quite different from $\mathsf{E}\mathbf{C}_j$ $(j=2,...,m-l+1)$ defined in Section 2. However, when $i>n+1$ and $j>m-l+1$, $\mathsf{E}\mathbf{C}^*_i=\mathsf{E}\mathbf{C}_j=\mathbf{\Gamma}$. Thus the condition $\lambda(\mathbf{\Gamma})<1$ is required for expressing $\mathsf{E}(u^2_tu^2_{t-n})$ as a function of $\mathsf{E}u^4_t$ and $\mathsf{E}u^2_t$.

Moreover, results of Lemma 5 make it possible to prove that $M_2(n)$ can be expressed as a function of $M_2(1),...,M_2(p)$ for $n\geq p+1$. We have

Lemma 6 . *Define the mixed moment $\mathsf{E}\left(u^2_tu^2_{t-n}\right)$ as in (43) and let $n>p$. Then, for $n\geq p+1$,*

$$M_2(n)=\boldsymbol{\gamma}'_{p+1}\mathbf{\Gamma}_*^{n-(p+1)}\mathbf{m}_2 \quad (50)$$

where $\mathbf{m}_2 = (0, M_2(p), ..., M_2(1))'$ *is a* $(p+1) \times 1$ *vector and* $M_2(i)$, $i = 1, ..., p$, *are given by (44)-(48).*

Proof. See Appendix 8.

It now follows from Lemma 1 that

$$\begin{aligned}\mathsf{E}\left(u_t^2 u_{t-n}^2\right) &= \mathsf{E}\left(\varepsilon_t^2 \varepsilon_{t-n}^2 S_{10}\right) + \mathsf{E}\left(\varepsilon_t^2 \varepsilon_{t-n}^2 S_{20}\right) + \sum_{i=n+1}^{k} \mathsf{E}\left(\varepsilon_t^2 \varepsilon_{t-n}^2 S_{1i}\right) \\ &\quad + \sum_{i=n+1}^{k} \mathsf{E}\left(\varepsilon_t^2 \varepsilon_{t-n}^2 S_{2i}\right) + \mathsf{E}\left(\varepsilon_t^2 \varepsilon_{t-n}^2 S_k\right). \qquad (51)\end{aligned}$$

Assume again that the process started infinitely far in the past at some finite value. Then, given the previous results, the limit of (51) as $k \to \infty$ exists and is independent of t if and only if all the eigenvalues of $\mathbf{\Gamma}$ lie inside the unit circle. This is obvious because it was seen that $\mathsf{E}u_t^4 < \infty$ requires $\lambda(\mathbf{\Gamma}) < 1$. As $k \to \infty$, the last three terms on the right-hand side of (51) can be evaluated by Lemmata 2 to 4. As for the first two terms, when $n \leq p$ we may apply Theorem 1, otherwise we apply Lemmata 5 and 6. The next theorem gives the moments.

Theorem 3. *The mixed moment* $\mathsf{E}(u_t^2 u_{t-n}^2)$ *has the form*

$$\mathsf{E}\left(u_t^2 u_{t-n}^2\right) = \alpha_0 \nu_2^2 M_1(n)\, \mathsf{E}h_t + \nu_2 M_2(n)\, \mathsf{E}h_t^2 \qquad (52)$$

where for $n \geq 1$,

$$M_1(n) = \boldsymbol{\gamma}_{p+1}' \mathbf{\Gamma}_*^{n-1} \left[\mathbf{e}_{\alpha_0} + \mathbf{\Gamma}_{n+1}^* \left(\mathbf{I}_{p^*} - \mathbf{\Gamma}\right)^{-1} \mathbf{e}_{p-1}\right] \qquad (53)$$

with $\mathbf{e}_{\alpha_0} = \left(\alpha_0^{-1}, 0, 1, ..., 1\right)'$ *is a* $(p+1) \times 1$ *vector.* $M_2(n)$ *in (52) is defined by (44)-(48) for* $n \leq p$*; otherwise* $M_2(n)$ *is given by (50).*

Proof. See Appendix 9.

3.2 The autocorrelation function for the squared process

Next we derive the autocorrelation function of $\{u_t^2\}$ and begin by introducing some notation. Let $\gamma_{M_1}(l,m) = (1-\Delta)M_1(l,m)$ and $\gamma_{M_2}(l,m) = (1-\Delta)M_2(l,m)$, where $1-\Delta = |\mathbf{I}_{p^*} - \mathbf{\Gamma}|$ is the determinant of $(\mathbf{I}_{p^*} - \mathbf{\Gamma})$. Furthermore, let

$$\gamma_{S_1} = (1+\gamma_1)(1-\Delta) + 2\sum_{l<m} \gamma_{l1}\gamma_{m1}\gamma_{M_1}(l,m) \tag{54}$$

$$\gamma_{S_2} = (1-\gamma_2)(1-\Delta) - 2\sum_{l<m} \gamma_{l1}\gamma_{M_2}(l,m). \tag{55}$$

A straightforward calculation shows that condition (33) is equivalent to

$$0 < \gamma_{S_2} < 1. \tag{56}$$

Applying Theorem 2 and Theorem 3 to equation (42) gives

Theorem 4. *Assume that condition (56) holds. For the GARCH(p,p) model (1) with (3), the autocorrelation function for $\{u_t^2\}$ is, for any $n \geq 1$,*

$$\rho_n = \frac{\nu_2\gamma_{S_1}(1-\gamma_1)M_2(n) - \nu_2^2\gamma_{S_2}[1-(1-\gamma_1)M_1(n)]}{\nu_4\gamma_{S_1}(1-\gamma_1) - \nu_2^2\gamma_{S_2}} \tag{57}$$

where $M_1(n)$ and $M_2(n)$ are defined in Theorem 3 and $M_1(l,m)$ and $M_2(l,m)$ as in Theorem 1.

Properties of the autocorrelation function $\{\rho_n\}$, $n = 1, 2, \cdots$, can be established through (57). Some of them are listed below:

1. The first p autocorrelations are positive if the parameter restrictions $\alpha_0 > 0$, $\alpha_i \geq 0$ and $\beta_i \geq 0$, $i = 1, \cdots, p$, hold.

2. The autocorrelation function satisfies the difference equation $\rho_n = \sum_{i=1}^{p} \gamma_{i1}\rho_{n-i}$ at lags $n > p$, and $\rho_n > 0$.

3. The autocorrelation function is dominated by an exponential decay and $\lim_{n\to\infty} \rho_n = 0$.

4. When γ_{S_2} is sufficiently close to zero the autocorrelation function is persistent; otherwise the autocorrelations decay quickly with increasing n.

3.3 Special case: GARCH$(2,2)$

To illustrate the general result we again consider the GARCH(2,2) process. In the GARCH(2,2) model,

$$\begin{aligned} \gamma_i &= \gamma_{1i} + \gamma_{2i},\ i = 1,2 \\ \gamma_{S_1} &= (1+\gamma_1)(1-\gamma_{21}) + 2\gamma_{11}\gamma_{21} \\ \gamma_{S_2} &= (1-\gamma_2)(1-\gamma_{21}) - 2\gamma_{11}\widetilde{\gamma}_{12}. \end{aligned}$$

By Theorem 4 we have

Corollary 3. *Assume that condition (36) holds. For the GARCH$(2,2)$ process, the autocorrelations of $\{u_t^2\}$ equal*

$$\rho_n = \frac{\nu_2\gamma_{S_1}(1-\gamma_1) M_2(n) - \nu_2^2\gamma_{S_2}[1-(1-\gamma_1) M_1(n)]}{\nu_4\gamma_{S_1}(1-\gamma_1) - \nu_2^2\gamma_{S_2}},\ n \geq 1. \tag{58}$$

In (58), for $n \geq 1$

$$M_1(n) = \boldsymbol{\gamma}_3'\boldsymbol{\Gamma}_*^{n-1}\begin{pmatrix} \alpha_0^{-1} \\ 0 \\ (1-\gamma_{21})^{-1} \end{pmatrix} \tag{59}$$

and for $n \geq 3$

$$M_2(n) = \boldsymbol{\gamma}_3'\boldsymbol{\Gamma}_*^{n-3}\begin{pmatrix} 0 \\ M_2(2) \\ M_2(1) \end{pmatrix} \tag{60}$$

with $\boldsymbol{\gamma}_3 = \begin{pmatrix} \alpha_0 & \gamma_{11} & \gamma_{21} \end{pmatrix}'$ *and* $\boldsymbol{\Gamma}_* = \begin{pmatrix} 1 & 0 & 0 \\ \alpha_0 & \gamma_{11} & \gamma_{21} \\ 0 & 1 & 0 \end{pmatrix}$, *whereas*

$$M_2(1) = \frac{1}{1-\gamma_{21}} \left[\overline{\gamma}_{11} (1-\gamma_{21}) + \nu_2 \widetilde{\gamma}_{12} \right], \tag{61}$$

$$M_2(2) = \frac{1}{1-\gamma_{21}} \left[(\gamma_{11}\overline{\gamma}_{11} + \overline{\gamma}_{21}) (1-\gamma_{21}) + \nu_2 \gamma_{11} \widetilde{\gamma}_{12} \right]. \tag{62}$$

Proof. See Appendix 10.

From Corollary 3 it follows that the first two autocorrelations are

$$\rho_1 = \frac{\nu_2 \gamma_{S_1} \left[\overline{\gamma}_{11} (1-\gamma_{21}) + \nu_2 \widetilde{\gamma}_{12} \right] (1-\gamma_1) - \nu_2^2 \gamma_{11} \gamma_{S_2}}{(1-\gamma_{21}) \left[\nu_4 \gamma_{S_1} (1-\gamma_1) - \nu_2^2 \gamma_{S_2} \right]} \tag{63}$$

and

$$\rho_2 = \frac{\nu_2 \gamma_{S_1} \left[(\gamma_{11}\overline{\gamma}_{11} + \overline{\gamma}_{21})(1-\gamma_{21}) + \nu_2 \gamma_{11} \widetilde{\gamma}_{12} \right] (1-\gamma_1) - \nu_2^2 \gamma_{S_2} \left(\gamma_{11}^2 + \gamma_{21} - \gamma_{21}^2 \right)}{(1-\gamma_{21}) \left[\nu_4 \gamma_{S_1} (1-\gamma_1) - \nu_2^2 \gamma_{S_2} \right]}. \tag{64}$$

For higher orders, ρ_n is determined by the powers of matrix $\boldsymbol{\Gamma}_*$ as well as $M_2(1)$ and $M_2(2)$ through equation (58). In other words, for $n \geq 3$, ρ_n depends on the first two autocorrelations, which is also clear from the corresponding Yule-Walker equations.

The corresponding results for GARCH(1,1) model are obtained by setting $\gamma_{21} = \widetilde{\gamma}_{12} = \overline{\gamma}_{21} = 0$ in (58). The autocorrelation function thus has the form

$$\rho_n = \frac{\nu_2 \left(1-\gamma_{11}^2\right) M_2(n) - \nu_2^2 (1-\gamma_{12}) \left[1 - (1-\gamma_{11}) M_1(n) \right]}{\nu_4 (1-\gamma_{11}^2) - \nu_2^2 (1-\gamma_{12})}, n \geq 1. \tag{65}$$

$M_1(n)$ and $M_2(n)$ can be determined as follows. We have $\boldsymbol{\Gamma}_* = \begin{pmatrix} 1 & 0 \\ \alpha_0 & \gamma_{11} \end{pmatrix}$. Thus, for $n \geq 1$,

$$\begin{aligned} M_1(n) &= \begin{pmatrix} \alpha_0 & \gamma_{11} \end{pmatrix} \boldsymbol{\Gamma}_*^{n-1} \begin{pmatrix} \alpha_0^{-1} \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha_0 & \gamma_{11} \end{pmatrix} \begin{pmatrix} \alpha_0^{-1} \\ \sum_{i=0}^{n-2} \left(\prod_{j=0}^{i} \gamma_{11}^j \right) \end{pmatrix} \\ &= \frac{1-\gamma_{11}^n}{1-\gamma_{11}}. \end{aligned} \tag{66}$$

Furthermore, for $n \geq 3$,

$$\begin{aligned} M_2(n) &= \begin{pmatrix} \alpha_0 & \gamma_{11} \end{pmatrix} \mathbf{\Gamma}_*^{n-3} \begin{pmatrix} 0 \\ \gamma_{11}\overline{\gamma}_{11} \end{pmatrix} = \begin{pmatrix} \alpha_0 & \gamma_{11} \end{pmatrix} \begin{pmatrix} 0 \\ \gamma_{11}^{n-2}\overline{\gamma}_{11} \end{pmatrix} \\ &= \gamma_{11}^{n-1}\overline{\gamma}_{11} \end{aligned} \tag{67}$$

with $M_2(1) = \overline{\gamma}_{11}$ and $M_2(2) = \gamma_{11}\overline{\gamma}_{11}$. In fact, $M_2(n) = \gamma_{11}^{n-1}\overline{\gamma}_{11}$ for $n \geq 1$. Inserting (66) and (67) into (65) yields

$$\begin{aligned} \rho_n &= \frac{\nu_2\gamma_{11}^{n-1}[\overline{\gamma}_{11}\left(1-\gamma_{11}^2\right) - \nu_2\gamma_{11}\left(1-\gamma_{12}\right)]}{\nu_4\left(1-\gamma_{11}^2\right) - \nu_2^2\left(1-\gamma_{12}\right)} \\ &= \frac{\alpha_1\nu_2\gamma_{11}^{n-1}(1-\beta_1^2-\alpha_1\beta_1\nu_2)}{1-\beta_1^2-2\alpha_1\beta_1\nu_2} \end{aligned} \tag{68}$$

see Teräsvirta (1996). Setting $\nu_2 = 1$ and $\nu_4 = 3$ in (68) yields the corresponding result for normal errors given in Bollerslev (1988).

3.4 The ARCH(p) process

As a further illustration, we reproduce the autocorrelation function of $\{u_t^2\}$ in an ARCH(p) model and compare our result with that of Milhϕj (1985); see (39) and (40). In an ARCH(p) model, $\beta_i = 0$ $i = 1, ..., p$. The autocorrelations thus have the form

$$\rho_n = \frac{\nu_2\gamma_{S_1}\left(1-\nu_2\sum_{i=1}^{p}\alpha_i\right)M_2(n) - \nu_2^2\gamma_{S_2}\left[1-\left(1-\nu_2\sum_{i=1}^{p}\alpha_i\right)M_1(n)\right]}{\nu_4\gamma_{S_1}\left(1-\nu_2\sum_{i=1}^{p}\alpha_i\right) - \nu_2^2\gamma_{S_2}} \tag{69}$$

for $n \geq 1$. Further simplification of (69) gives

Corollary 4. *For the ARCH(p) model, the autocorrelation function of $\{u_t^2\}$ has the form*

$$\rho_n = 1 - \left(1-\nu_2\sum_{i=1}^{p}\alpha_i\right)M_1(n),\ n \geq 1 \tag{70}$$

where

$$M_1(n) = \boldsymbol{\gamma}_{p+1}^{'}\mathbf{\Gamma}_*^{n-1}\left[\mathbf{e}_{\alpha_0} + \mathbf{\Gamma}_{n+1}^{*}\left(\mathbf{I}_{p^*} - \mathbf{\Gamma}\right)^{-1}\mathbf{e}_{p-1}\right]. \tag{71}$$

Proof. See Appendix 11.

If the errors are assumed normal, then (70) gives the corresponding result which is the one Milhϕj (1985) obtained. As an example, for the ARCH(2) model it follows from (70), or (39) and (40), that $\rho_1 = \alpha_1(1-\alpha_2)^{-1}$, $\rho_2 = \alpha_2+\alpha_1^2(1-\alpha_2)^{-1}$, and $\rho_n = \alpha_1\rho_{n-1}+\alpha_2\rho_{n-2}$ for $n \geq 2$. Alternatively, a general expression for ρ_n, $n \geq 2$, is obtained from Corollary 4.

Milhϕj (1985) also found a necessary and sufficient condition for the existence of the unconditional fourth moment of the ARCH(p) process under normality. It has the form

$$3\boldsymbol{\alpha}^{'}\left(\mathbf{I}_p-\boldsymbol{\Psi}\right)^{-1}\boldsymbol{\alpha} < 1 \tag{72}$$

where $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_p)^{'}$ and the $p \times p$ matrix $\boldsymbol{\Psi} = \left(\psi_{ij}\right)$ is defined by $\psi_{ij} = \alpha_{i+j} + \alpha_{i-j}$ with $\alpha_k = 0$ for $k \leq 0$ and $k > p$. Of course, (33) under normality and (72) are equivalent, although this may not be easy to see immediately. As an illustration, we demonstrate this equivalence in the ARCH(2) case. Setting $\gamma_{i1} = \alpha_i, \gamma_{i2} = 3\alpha_i^2$, $i = 1, 2$, and $\widetilde{\gamma}_{ij} = 3\alpha_1\alpha_2$ in (36) yields

$$3\alpha_1^2 + 3\alpha_2^2 + 3\alpha_1^2\alpha_2 - 3\alpha_2^3 + \alpha_2 < 1. \tag{73}$$

On the other hand, setting $\boldsymbol{\alpha} = (\alpha_{1,}\alpha_2)'$ and $\boldsymbol{\Psi} = \begin{pmatrix} \alpha_2 & 0 \\ \alpha_1 & 0 \end{pmatrix}$ in (72) also yields (73).

4 The GARCH(p, q) model

The results in Theorems 1 to 4 apply directly to the GARCH(p, q) model with $p \neq q$. This is because we may, without any loss of generality, nest that model in a model with $p = q$ by assuming certain parameters in a GARCH(p, p) or a GARCH(q, q) model equal to zero.

Theorem 5.

1. *Let $p > q$ in the GARCH(p, q) model. Then the necessary and sufficient condition*

for the existence of the fourth moment $\mathsf{E}u_t^4$ *and the autocorrelation function for* $\{u_t^2\}$ *are given in Theorems 2 and 4 for the GARCH*(p,p) *model. In that case,* $\gamma_{i1}, \gamma_{i2}, \overline{\gamma}_{i1}$ *and* $\widetilde{\gamma}_{ij}$ *simplify to*

$$\gamma_{i1} = \begin{cases} \gamma_{i1} & for \quad i = 1,...,q \\ \beta_i & for \quad i = q+1,...,p \end{cases}, \; \gamma_{i2} = \begin{cases} \gamma_{i2} & for \quad i = 1,...,q \\ \beta_i^2 & for \quad i = q+1,...,p \end{cases}$$

$$\overline{\gamma}_{i1} = \begin{cases} \overline{\gamma}_{i1} & for \quad i = 1,...,q \\ \beta_i \nu_2 & for \quad i = q+1,...,p \end{cases}, \; \widetilde{\gamma}_{ij} = \begin{cases} \widetilde{\gamma}_{ij} & for \quad i < j \le q \\ \beta_j \gamma_{i1} & for \quad i \le q < j \\ \beta_i \beta_j & for \quad q < i < j. \end{cases}$$

2. *Let* $p < q$ *in the GARCH*(p,q) *model. Then the necessary and sufficient condition for the existence of the fourth moment* $\mathsf{E}u_t^4$ *and the autocorrelation function* ρ_n *for* $\{u_t^2\}$ *are given in Theorems 2 and 4 for the GARCH*(q,q) *model. In that case,* $\gamma_{i1}, \gamma_{i2}, \overline{\gamma}_{i1}$ *and* $\widetilde{\gamma}_{ij}$ *simplify to*

$$\gamma_{i1} = \begin{cases} \gamma_{i1} & if \quad i = 1,...,p \\ \alpha_i \nu_2 & if \quad p+1,...,q \end{cases}, \; \gamma_{i2} = \begin{cases} \gamma_{i2} & if \quad i = 1,...,p \\ \alpha_i^2 \nu_4 & if \quad p+1,...,q \end{cases}$$

$$\overline{\gamma}_{i1} = \begin{cases} \overline{\gamma}_{i1} & if \quad i = 1,...,p \\ \alpha_i \nu_4 & if \quad p+1,...,q \end{cases}, \; \widetilde{\gamma}_{ij} = \begin{cases} \widetilde{\gamma}_{ij} & if \quad i < j \le p \\ \alpha_j \overline{\gamma}_{i1} & if \quad i \le p < j \\ \alpha_i \alpha_j \nu_4 & if \quad p < i < j \le q. \end{cases}$$

Examples of the use of this theorem can be found in He and Teräsvirta (1997).

5 Conclusions

We have obtained a complete characterization of the fourth moment structure of a general GARCH(p,q) process. With our results, an investigator can see what an estimated GARCH model implies about the second and the fourth moments, kurtosis, and the autocorrelation function of the centred and squared observations. Such considerations have previously been possible in the GARCH(1,1) case. These results can be extended to other GARCH processes which are generalizations of the original GARCH(p, q) process. For example, some GARCH processes allowing for asymmetric effects to shocks belong to this category. Those generalizations are a topic of further work.

We have not considered moments of higher than fourth order. Deriving those using the present techniques would no doubt be tedious. On the other hand, as underlined above, the fourth moments are probably more interesting in practice than any higher order ones.

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Appendices

Appendix 1. Proof of Lemma 1.

Let $1 \leq n \leq p$. Applying (3) to h_t in $h_t h_{t-n}$ yields

$$h_t h_{t-n} = \alpha_0 h_{t-n} + c_{n,t-n} h_{t-n}^2 + \mathbf{c}'_{P\backslash\{n\}} \mathbf{h}_{1t} h_{t-n}, \tag{A.1}$$

where $\mathbf{h}_{1t} = (h_{t-1}, \cdots, h_{t-n+1}, h_{t-n-1}, \cdots, h_{t-p})'$. Applying (3) to h_{t-1} on the right-hand side of (A.1) and continuing the iteration until the appearance of the matrix $\mathbf{C}_{n+1}$ defined by (8) gives

$$\begin{aligned} h_t h_{t-n} &= \alpha_0 h_{t-n} + c_{n,t-n} h_{t-n}^2 + \alpha_0 \mathbf{c}'_{P\backslash\{n\}} \sum_{i=1}^{n-1} \left(\prod_{j=1}^{i} \mathbf{C}_j \right) \mathbf{e}_1 h_{t-n} \\ &+ \mathbf{c}'_{P\backslash\{n\}} \sum_{i=1}^{n-1} \left(\prod_{j=1}^{i} \mathbf{C}_j \right) \mathbf{e}_1 c_{n-i,t-n} h_{t-n}^2 \\ &+ \alpha_0 \mathbf{c}'_{P\backslash\{n\}} \prod_{i=1}^{n} \mathbf{C}_i \mathbf{h}_{10t} + \mathbf{c}'_{P\backslash\{n\}} \prod_{i=1}^{n} \mathbf{C}_i \mathbf{h}_{20t} \\ &+ \mathbf{c}'_{P\backslash\{n\}} \prod_{i=1}^{n+1} \mathbf{C}_i \mathbf{h}_{n+1,t}, \end{aligned} \tag{A.2}$$

where $\mathbf{C}_1, ..., \mathbf{C}_{n+1}$ are given by (6)-(8) and $\mathbf{h}_{n+1,t}$ has the form

$$\begin{aligned} \mathbf{h}_{n+1,t} &= (h_{t-n-1} h_{t-n-2}, \cdots, h_{t-n-1} h_{t-n-p}, \\ &\quad h_{t-n-2} h_{t-n-3}, \cdots, h_{t-n-2} h_{t-n-p}, \cdots, h_{t-n-p+1} h_{t-n-p})'. \end{aligned} \tag{A.3}$$

In particular, $\mathbf{C}_{n+1}$ is a $(p-1) \times p^*$ matrix corresponding to the p^*-component column vector given in (A.3). When $i > n+1$, applying (3) to h_{t-i} in vector $\mathbf{h}_{it}$ on the left-hand side of (A.4) yields

$$\mathbf{h}_{it} = \alpha_0 \mathbf{h}_{1,i+1,t} + \mathbf{h}_{2,i+1,t} + \mathbf{C}_{i+1} \mathbf{h}_{i+1,t} \tag{A.4}$$

where $\mathbf{h}_{it}$, $\mathbf{h}_{i+1,t}$, $\mathbf{h}_{1,i+1,t}$ and $\mathbf{h}_{2,i+1,t}$ are defined in (17). Further recursions of (A.1) by applying (A.4) lead to equation (12).

Similarly, it can be shown that (12) and (14) hold when $n > p$. $\dashv$

Appendix 2. Proof of Lemma 2.

Substituting $t-l$ for t and $m-l$ for n in Lemma 1 we obtain a recursion formula for $h_{t-l}h_{t-m}$, $l<m$. This technique will also be used in Appendices 3-5. Let $k \geq p+m-l$. From (13),

$$\begin{aligned}
&\mathsf{E}\left(c_{l,t-l}c_{m,t-m}S_k\right) \\
&= \gamma_{l1}\gamma_{m1}\mathsf{E}(\mathbf{c}'_{P\setminus\{m-l\}}\mathbf{C}_1\cdots\mathbf{C}_{m-l+1})\mathsf{E}(\mathbf{C}_{m-l+2}\cdots\mathbf{C}_{k-p+1})\mathsf{E}(\mathbf{C}_{k-p+2}\cdots\mathbf{C}_k\mathbf{h}_{kt}). \quad (A.5)
\end{aligned}$$

First, when $k \geq p+m-l$, the product $\mathbf{C}_{k-p+2}\cdots\mathbf{C}_k\mathbf{h}_{kt}$ is not a function of k, so that the same is true for its mean. Second, for $i,j > m-l+1$, $i \neq j$, we have $\mathsf{E}\mathbf{C}_i\mathbf{C}_j = \mathsf{E}\mathbf{C}_i\mathsf{E}\mathbf{C}_j$. Thus, for any $k \geq p+m-l$,

$$\mathsf{E}\left(\mathbf{C}_{m-l+2}\cdots\mathbf{C}_{k-p+1}\right) = \mathbf{\Gamma}^{k-(p+m-l)}. \quad (A.6)$$

Furthermore,

$$\lim_{k\to\infty}\mathbf{\Gamma}^{k-(p+m-l)} = \mathbf{0} \quad (A.7)$$

if and only if $\lambda(\mathbf{\Gamma}) < 1$. (A.5) and (A.7) together imply that

$$\mathsf{E}\left(c_{l,t-l}c_{m,t-m}S_k\right) \to 0$$

as $k \to \infty$, if and only if $\lambda(\mathbf{\Gamma}) < 1$. ⊣

Appendix 3. Proof of Lemma 3.

By (13),

$$\begin{aligned}
&\mathsf{E}\left(c_{l,t-l}c_{m,t-m}\sum_{i=m-l+1}^{k}S_{1i}\right) \\
&= \gamma_{l1}\gamma_{m1}\sum_{i=m-l+1}^{k}\mathsf{E}\left(S_{1i}\right) \\
&= \alpha_0\gamma_{l1}\gamma_{m1}\boldsymbol{\gamma}'_{P\setminus\{m-l\}}\mathsf{E}\left(\mathbf{C}_1\cdots\mathbf{C}_{m-l+1}\right)\left[\sum_{i=m-l+1}^{k}\mathsf{E}\left(\mathbf{C}_{m-l+2}\cdots\mathbf{C}_i\right)\right]\mathsf{E}\left(\mathbf{h}_{1it}\right) \\
&= \alpha_0\gamma_{l1}\gamma_{m1}\boldsymbol{\gamma}'_{P\setminus\{m-l\}}\mathbf{\Gamma}_1\cdots\mathbf{\Gamma}_{m-l+1}\left(\sum_{i=m-l+1}^{k}\mathbf{\Gamma}^{i-(m-l+1)}\right)\mathbf{e}_{p-1}\mathsf{E}h_t. \quad (A.8)
\end{aligned}$$

Note that

$$\sum_{i=m-l+1}^{k} \mathbf{\Gamma}^{i-(m-l+1)} \rightarrow (\mathbf{I}_{p^*} - \mathbf{\Gamma})^{-1}$$

as $k \rightarrow \infty$, if and only if $\lambda(\mathbf{\Gamma}) < 1$. Thus (21) is valid. ⊣

Appendix 4. Proof of Lemma 4.

Suppose that $k > m - l + p$. By (13),

$$\begin{aligned}
& \mathsf{E}\left(c_{l,t-l} c_{m,t-m} \sum_{i=m-l+1}^{k} S_{2i}\right) \\
= \; & \gamma_{l1}\gamma_{m1} \sum_{i=m-l+1}^{m-l+p-1} \mathsf{E}(S_{2i}) + \gamma_{l1}\gamma_{m1} \sum_{i=m-l+p}^{k} \mathsf{E}(S_{2i}) \\
= \; & \gamma_{l1}\gamma_{m1} \sum_{i=m-l+1}^{m-l+p-1} \mathsf{E}(S_{2i}) + \gamma_{l1}\gamma_{m1} \boldsymbol{\gamma}'_{P\setminus\{m-l\}} \prod_{j=1}^{m-l+1} \mathbf{\Gamma}_j \\
& \times \left[\sum_{i=m-l+p}^{k} \mathsf{E}(\mathbf{C}_{m-l+2} \cdots \mathbf{C}_{i-p+1}) \mathsf{E}(\mathbf{C}_{i-p+2} \cdots \mathbf{C}_i \mathbf{h}_{2it})\right]. \qquad \text{(A.9)}
\end{aligned}$$

Arguing as in Appendix 2, $\mathsf{E}(\mathbf{C}_{i-p+2} \cdots \mathbf{C}_i \mathbf{h}_{2it})$ does not depend on i for $i > m - l + 2p - 2$. Setting $i = m - l + p$ in (22) gives

$$\mathsf{E}(\mathbf{C}_{m-l+p+1} \cdots \mathbf{C}_{m-l+2p-1} \mathbf{c}_{2,m-l+p,t}) = \boldsymbol{\gamma}(m - l + p + 1, m - l + 2p - 1).$$

Then, for any $i > m - l + 2p - 2$,

$$\mathsf{E}(\mathbf{C}_{i-p+2} \cdots \mathbf{C}_i \mathbf{c}_{2it}) = \boldsymbol{\gamma}(m - l + p + 1, m - l + 2p - 1). \qquad \text{(A.10)}$$

On the other hand, by (A.6),

$$\sum_{i=m-l+p}^{k} \mathsf{E}\left(\mathbf{C}_{m-l+p+1} \cdots \mathbf{C}_{i-(p+1)}\right) = \sum_{i=m-l+p}^{k} \mathbf{\Gamma}^{i-(m-l+p)}$$

which implies

$$\lim_{k\rightarrow\infty} \sum_{i=m-l+p}^{k} \mathbf{\Gamma}^{i-(m-l+p)} = (\mathbf{I}_{p_*} - \mathbf{\Gamma})^{-1} \qquad \text{(A.11)}$$

if and only if $\lambda(\mathbf{\Gamma}) < 1$. Under this condition and applying (A.10), (A.11) and (23) to (A.9) we see that equation (24) holds. ⊣

Appendix 5. Proof of Theorem 1.

We shall show that (25)-(31) hold. From (13) we obtain that

$$\begin{aligned}&\mathsf{E}\left(c_{l,t-l}c_{m,t-m}S_{10}\right)\\ =\;& \alpha_0\gamma_{l1}\gamma_{m1}\left[1+\boldsymbol{\gamma}'_{P\backslash\{m-l\}}\sum_{i=1}^{m-l-1}\left(\prod_{j=1}^{i}\boldsymbol{\Gamma}_j\right)\mathbf{e}_1+\prod_{i=1}^{m-l}\boldsymbol{\Gamma}_i\mathbf{j}_{p-1}\right]\mathsf{E}h_t \qquad \text{(A.12)}\end{aligned}$$

provided $\mathbf{C}_{m-l}$ is not an identity matrix. Following Lemma 3 and equation (A.12) we may define the coefficients of $\mathsf{E}h_t$ for the sum of $\mathsf{E}\left(c_{l,t-l}c_{m,t-m}S_{10}\right)$ and $\mathsf{E}\left(c_{l,t-l}c_{m,t-m}\sum_{i=m-l+1}^{\infty}S_{1i}\right)$ as follows. When $m-l>1$,

$$M_1(l,m)=1+\boldsymbol{\gamma}'_{P\backslash\{m-l\}}\left[\sum_{i=1}^{m-l-1}\left(\prod_{j=1}^{i}\boldsymbol{\Gamma}_j\right)\mathbf{e}_1+\prod_{i=1}^{m-l}\boldsymbol{\Gamma}_i\left(\mathbf{j}_{p-1}+\boldsymbol{\Gamma}_{m-l+1}\left(\mathbf{I}_{p^*}-\boldsymbol{\Gamma}\right)^{-1}\mathbf{e}_{p-1}\right)\right] \qquad \text{(A.13)}$$

and when $m-l=1$, (A.13) simplifies to

$$M_1(m-1,m)=1+\boldsymbol{\gamma}'_{P\backslash\{1\}}\left[\mathbf{j}_{p-1}+\boldsymbol{\Gamma}_2\left(\mathbf{I}_{p^*}-\boldsymbol{\Gamma}\right)^{-1}\mathbf{e}_{p-1}\right] \qquad \text{(A.14)}$$

since $\boldsymbol{\Gamma}_1=\mathbf{I}_{p-1}$. Equation (26) is thus valid.

To evaluate $\mathsf{E}\left(c_{l,t-l}c_{m,t-m}S_{20}\right)$, consider first

$$\mathsf{E}\left(c_{l,t-l}c_{m,t-m}c_{m-l,t-m}h^2_{t-m}\right)=\gamma_{l1}\widetilde{\gamma}_{m-l,m}\mathsf{E}h^2_t. \qquad \text{(A.15)}$$

Second, when $m-l>1$,

$$\begin{aligned}&\mathsf{E}\left[c_{l,t-l}c_{m,t-m}\mathbf{c}'_{P\backslash\{m-l\}}\sum_{i=1}^{m-l-1}\left(\prod_{j=1}^{i}\mathbf{C}_j\right)\mathbf{e}_1c_{m-l-i,t-m}h^2_{t-m}\right]\\ =\;&\left[\gamma_{l1}\boldsymbol{\gamma}'_{P\backslash\{m-l\}}\sum_{i=2}^{m-l}\left(\prod_{j=2}^{i}\boldsymbol{\Gamma}_j\right)\mathbf{e}_1\widetilde{\gamma}_{m-l-i,m}\right]\mathsf{E}h^2_t. \qquad \text{(A.16)}\end{aligned}$$

Combining (A.15) and (A.16) and observing (25) yields (28). Finally, if $m-l=1$, then

$$\mathsf{E}\left(c_{l,t-l}c_{m,t-m}\mathbf{c}'_{P\backslash\{1\}}\mathbf{h}_{20t}\right)=\gamma_{l1}\gamma_{m1}\left(\sum_{j=2}^{p}\widetilde{\gamma}_{j-1,j}\right)\mathsf{E}h^2_t. \qquad \text{(A.17)}$$

For $m-l=2$,

$$\begin{aligned}&\mathsf{E}\left(c_{l,t-l}c_{m,t-m}\mathbf{c}'_{P\backslash\{2\}}\mathbf{C}_1\mathbf{C}_2\mathbf{h}_{20t}\right)\\ =\ &\gamma_{l1}\gamma_{m1}\left(\gamma_{11}\sum_{j=2}^{p}\widetilde{\gamma}_{j-1,j}+\sum_{j=3}^{p}\widetilde{\gamma}_{j-2,j}\right)\mathsf{E}h_t^2. \end{aligned}\qquad \text{(A.18)}$$

Let $M_{22}(1)=M_{22}(m-1,m)=\sum_{j=2}^{p}\widetilde{\gamma}_{j-1,j}$. We may then write the coefficients of $\mathsf{E}h_t^2$ in equation (A.18) as

$$M_{22}(2)=M_{22}(m-2,m)=\gamma_{11}M_{22}(1)+\sum_{j=3}^{p}\widetilde{\gamma}_{j-2,j}. \qquad \text{(A.19)}$$

Continuing the recursion for $m-l=3,...,p-1$ proves (29).

Next, (30) and (31) follow from Lemma 4. Letting $k\to\infty$ in (19) gives

$$\mathsf{E}\left(c_{l,t-l}c_{m,t-m}h_{t-l}h_{t-m}\right)=\alpha_0\gamma_{l1}\gamma_{m1}M_1(l,m)\mathsf{E}h_t+\gamma_{l1}\left[M_{21}(l,m)+\gamma_{m1}\sum_{i=2}^{4}M_{2i}(l,m)\right]\mathsf{E}h_t^2 \qquad \text{(A.20)}$$

which, given definition (27), equals (25). This completes the proof. ⊣

Appendix 6. Proof of Corollary 1.

In the GARCH(2,2) model, $\mathbf{\Gamma}=\gamma_{21}$. Thus (35) holds if and only if $\gamma_{21}<1$. Note that $m-l=1$ and $\mathbf{\Gamma}_2=\gamma_{21}$. By (26),

$$M_1(1,2)=1+\gamma_{21}\left[1+\gamma_{21}(1-\gamma_{21})^{-1}\right]=(1-\gamma_{21})^{-1}.$$

Furthermore, by (28)-(31),

$$\begin{aligned}M_{21}(1,2)&=M_{22}(1,2)=\widetilde{\gamma}_{12}\\ M_{23}(1,2)&=\gamma(\mathbf{c}(1),2)=\mathsf{E}\left(c_{2,t-3}c_{2,t-4}c_{1,t-4}\right)=\gamma_{21}\widetilde{\gamma}_{12}\\ M_{24}(1,2)&=\gamma_{21}\mathbf{\Gamma}_1\mathbf{\Gamma}_2(\mathbf{I}_1-\mathbf{\Gamma})^{-1}\widetilde{\gamma}_{12}=\gamma_{21}^2\widetilde{\gamma}_{12}(1-\gamma_{21})^{-1}.\end{aligned}$$

Finally, from (27),

$$M_2(1,2)=\widetilde{\gamma}_{12}(1-\gamma_{21})^{-1}. \qquad ⊣$$

Appendix 7. Proof of Lemma 5.

Define the $(p+1)\times 1$ vector

$$\mathbf{h}(n) = \left(0, \mathsf{E}\left(\varepsilon_{t-n}^2 h_2(n)\right), \cdots, \mathsf{E}\left(\varepsilon_{t-n}^2 h_{p+1}(n)\right)\right).$$

First, we show that for $n > p$,

$$\mathsf{E}\left(\varepsilon_{t-n}^2 \mathbf{C}_{n-(p-1)}^* \mathbf{C}_{n-(p-2)}^* \cdots \mathbf{C}_n^* \mathbf{c}_{20t}^*\right) = \mathbf{h}(n) \tag{A.21}$$

where, for $i = 2, ..., p+1$,

$$h_i(n) = \sum_{j=2}^{p+1} \phi_{ij}(n)\, c_{j-2,t-(n+j-2)}. \tag{A.22}$$

In (A.22), $c_{0,t-n} = 1$, and $\phi_{ij}(n)$ are determined recursively with respect to integers $m = n-(p-1), n-(p-2), ..., n$, such that

$$\phi_{ij}(m) = \begin{cases} \phi_{i2}(m-1)\, c_{j-1,t-(m-j+2)} + \phi_{i,j+1}(m-1) & \text{for } j = 2, ..., p \\ \phi_{i2}(m-1)\, c_{p,t-(m+p-1)} & \text{for } j = p+1 \end{cases} \tag{A.23}$$

with the initial values $\phi_{ij}(n-(p-1)) = c_{ij}$ for any i and j, where c_{ij} is the (i,j)th element of matrix $\mathbf{C}_{n-(p-1)}^*$.

Let $\prod_{m=n-(p-1)}^{n} \mathbf{C}_m^* = \mathbf{\Phi}_n$, where $\mathbf{C}_m^*$ are given by (10). Since each $\mathbf{C}_m^*$ has the same first row $(1, 0, ..., 0)$, we can define

$$\mathbf{\Phi}_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \phi_{21}(n) & \phi_{22}(n) & \cdots & \phi_{2,p+1}(n) \\ \vdots & \vdots & & \vdots \\ \phi_{p+1,1}(n) & \phi_{p+1,2}(n) & \cdots & \phi_{p+1,p+1}(n) \end{pmatrix} \tag{A.24}$$

and rewrite (A.24) as

$$\mathbf{\Phi}_n = \mathbf{\Phi}_{n-1} \mathbf{C}_n^*. \tag{A.25}$$

Equalities between the elements of the matrices on both sides of (A.25) appear in (A.23). We see that $\mathsf{E}(\varepsilon_{t-n}^2 \mathbf{\Phi}_n \mathbf{c}_{20t}^*)$ can be expressed in terms of $h_i(n)$ defined by (A.22) and (A.23).

Next, let $n = p + 1$. Then by (A.21),

$$\mathsf{E}\left(\varepsilon_{t-(p+1)}^2 \mathbf{C}_2^* \mathbf{C}_3^* \cdots \mathbf{C}_{p+1}^* \mathbf{c}_{20t}^*\right) = \mathbf{h}(p+1) \tag{A.26}$$

where $\mathbf{c}_{20t}^* = (0, 1, c_{1,t-(p+2)}, \cdots, c_{p-1,t-2p})'$. For any $n \geq p + 1$, it follows from (A.21) with (A.22) and (A.23) that

$$\mathsf{E}\left(\varepsilon_{t-n}^2 \mathbf{C}_{n-(p-1)}^* \mathbf{C}_{n-(p-2)}^* \cdots \mathbf{C}_n^* \mathbf{c}_{20t}^*\right) = \mathbf{h}(p+1). \tag{A.27}$$

Finally, without loss of generality, we merely show that (49) is true for $p = 2$. From (A.26),

$$\mathsf{E}\left(\varepsilon_{t-3}^2 \mathbf{C}_2^* \mathbf{C}_3^* \begin{pmatrix} 0 \\ 1 \\ c_{1,t-4} \end{pmatrix}\right) = \begin{pmatrix} 0 \\ \gamma_{11}(\overline{\gamma}_{11} + \nu_2 \widetilde{\gamma}_{12}) + \overline{\gamma}_{21} \\ \overline{\gamma}_{11} + \nu_2 \widetilde{\gamma}_{12} \end{pmatrix} \tag{A.28}$$

where $\overline{\gamma}_{11} = \mathsf{E}(\varepsilon_t^2 c_{1t}), \overline{\gamma}_{21} = \mathsf{E}(\varepsilon_t^2 c_{2t})$ and $\widetilde{\gamma}_{12} = \mathsf{E}(c_{1t} c_{2t})$. On the other hand, by equations (45) and (46),

$$\begin{aligned} M_{21}(1) + \nu_2 M_{22}(1) &= \overline{\gamma}_{11} + \nu_2 \widetilde{\gamma}_{12}, \\ M_{21}(2) + \nu_2 M_{22}(2) &= \overline{\gamma}_{21} + \gamma_{11}(\overline{\gamma}_{11} + \nu_2 \widetilde{\gamma}_{12}). \quad \dashv \end{aligned} \tag{A.29}$$

Appendix 8. Proof of Lemma 6**.**

First, for any $n > p$, we show that the coefficient of $\mathsf{E}h_t^2$ in (43) is a function of n. By (14),

$$\begin{aligned} &\sum_{i=n+1}^{\infty} \mathsf{E}\left(\varepsilon_t^2 \varepsilon_{t-n}^2 S_{2i}\right) \\ =\ & \nu_2^2 \sum_{i=n+1}^{n+p-1} \boldsymbol{\gamma}(\mathbf{c}_{p+1}, 2, i)\, \mathsf{E}h_t^2 \\ & + \nu_2^2 \boldsymbol{\gamma}_{p+1}' \mathbf{\Gamma}_*^{n-1} \mathbf{\Gamma}_{n+1}^* (\mathbf{I}_{p^*} - \mathbf{\Gamma})^{-1} \boldsymbol{\gamma}(n+p, n+2p-2)\, \mathsf{E}h_t^2 \end{aligned} \tag{A.30}$$

Set $n = p+1$. Then (A.30) simplifies to

$$\begin{aligned}
&\sum_{i=p+2}^{\infty} \mathsf{E}\left(\varepsilon_t^2 \varepsilon_{t-(p+1)}^2 S_{2i}\right) \\
= \;& \nu_2^2 \boldsymbol{\gamma}_{p+1}' \boldsymbol{\Gamma}_*^2 \sum_{i=1}^{p-1} \boldsymbol{\Gamma}_*^{i-1} \boldsymbol{\gamma}\left(3+i, p+1+i\right) \mathsf{E} h_t^2 \\
&+ \nu_2^2 \boldsymbol{\gamma}_{p+1}' \boldsymbol{\Gamma}_*^{n-1} \boldsymbol{\Gamma}_{n+1}^* \left(\mathbf{I}_{p^*} - \boldsymbol{\Gamma}\right)^{-1} \boldsymbol{\gamma}\left(2p+1, 3p-1\right) \mathsf{E} h_t^2. \qquad \text{(A.31)}
\end{aligned}$$

For any $n > p$, $\boldsymbol{\gamma}\left(2p+1, 3p-1\right)$ in (A.31) remains unchanged. Thus

$$\begin{aligned}
&\sum_{i=n+1}^{\infty} \mathsf{E}\left(\varepsilon_t^2 \varepsilon_{t-n}^2 S_{2i}\right) \\
= \;& \nu_2^2 \boldsymbol{\gamma}_{p+1}' \boldsymbol{\Gamma}_*^{n-(p-1)} \sum_{i=1}^{p-1} \boldsymbol{\Gamma}_*^{i-1} \boldsymbol{\gamma}\left(3+i, p+1+i\right) \mathsf{E} h_t^2 \\
&+ \nu_2^2 \boldsymbol{\gamma}_{p+1}' \boldsymbol{\Gamma}_*^{n-1} \boldsymbol{\Gamma}_{n+1}^* \left(\mathbf{I}_{p^*} - \boldsymbol{\Gamma}\right)^{-1} \boldsymbol{\gamma}\left(2p+1, 3p-1\right) \mathsf{E} h_t^2 \\
= \;& \nu_2^2 M_{23}(n) + \nu_2^2 M_{24}(n), \;\; n > p. \qquad \text{(A.32)}
\end{aligned}$$

Furthermore, by (14) and Lemma 5 we can write, for any $n > p$,

$$\begin{aligned}
\mathsf{E}(\varepsilon_{t-n}^2 S_{20}) &= (\boldsymbol{\gamma}_{p+1}' \boldsymbol{\Gamma}_*^{n-(p+1)} \mathbf{m}_1) \mathsf{E} h_t^2 \\
&= (M_{21}(n) + \nu_2 M_{22}(n)) \mathsf{E} h_t^2 \qquad \text{(A.33)}
\end{aligned}$$

where $\mathbf{m}_1 = (0, M_{21}(p) + \nu_2 M_{22}(p), \cdots, M_{21}(1) + \nu_2 M_{22}(1))'$. It follows from (A.32) and (A.33) that $M_2(n) = M_{21}(n) + \nu_2 \sum_{i=2}^{4} M_{2i}(n)$ holds for any $n > p$.

It remains to show that (50) holds for any $n > p$. Without loss of generality, we prove that it holds for $p = 2$. For proof, see Appendix 10. $\dashv$

Appendix 9. Proof of Theorem 3.

Note that only the second column of $\boldsymbol{\Phi}_n = \prod_{i=n-(p-1)}^{n} \mathbf{C}_i^*$ depends on variables ε_{t-n}^2. Thus,

$$\mathsf{E}\left(\varepsilon_t^2 \varepsilon_{t-n}^2 \mathbf{c}_{p+1}^{*\prime} \mathbf{C}_1^* \cdots \mathbf{C}_n^* \mathbf{h}_{10t}^*\right) = \left(\alpha_0 \nu_2^2 \boldsymbol{\gamma}_{p+1}' \boldsymbol{\Gamma}_*^{n-1} \mathbf{e}_{\alpha_0}\right) \mathsf{E} h_t \qquad \text{(A.34)}$$

and

$$\sum_{i=n+1}^{\infty} \mathsf{E}\left(\varepsilon_t^2 \varepsilon_{t-n}^2 S_{1i}\right) = \left(\alpha_0 \nu_2^2 \boldsymbol{\gamma}_{p+1}' \boldsymbol{\Gamma}_*^{n-1} \boldsymbol{\Gamma}_{n+1}^* \left(\mathbf{I}_{p^*} - \boldsymbol{\Gamma}\right)^{-1} \mathbf{e}_{p-1}\right) \mathsf{E} h_t. \tag{A.35}$$

From (A.34) and (A.35) we see that (53) holds. ⊣

Appendix 10. Proof of Corollary 3.

Here we only illustrate how to calculate $M_1(n)$ and $M_2(n)$ for the GARCH(2,2) model.

Write

$$\boldsymbol{\Gamma}_* = \mathsf{E}\left(\mathbf{C}_i^*\right) = \begin{pmatrix} 1 & 0 & 0 \\ \alpha_0 & \gamma_{11} & \gamma_{12} \\ 0 & 1 & 0 \end{pmatrix}$$

for $i = 2, ..., n$, where $\mathbf{C}_i^*$ are defined by (10). By (53),

$$\begin{aligned} M_1(n) &= \boldsymbol{\gamma}_3' \boldsymbol{\Gamma}_*^{n-1} \left[\begin{pmatrix} \alpha_0^{-1} \\ 0 \\ 1 \end{pmatrix} + \frac{1}{1-\gamma_{21}} \begin{pmatrix} 0 \\ 0 \\ \gamma_{21} \end{pmatrix} \right] \\ &= \boldsymbol{\gamma}_3' \boldsymbol{\Gamma}_*^{n-1} \begin{pmatrix} \alpha_0^{-1} \\ 0 \\ (1-\gamma_{21})^{-1} \end{pmatrix} \end{aligned} \tag{A.36}$$

for $n = 1, 2, \cdots$. By (45)-(48),

$$\begin{aligned} M_{21}(1) &= \overline{\gamma}_{11} \\ M_{22}(1) &= \widetilde{\gamma}_{12} \\ M_{23}(1) &= \gamma\left(\mathbf{c}(1), 2\right) = \mathsf{E}\left(c_{2,t-3} c_{2,t-4} c_{1,t-4}\right) = \gamma_{21} \widetilde{\gamma}_{12} \\ M_{24}(1) &= \gamma_{21} \boldsymbol{\Gamma}_2 \frac{1}{1-\gamma_{21}} \widetilde{\gamma}_{12} = \gamma_{21}^2 \widetilde{\gamma}_{12} (1-\gamma_{21})^{-1}. \end{aligned}$$

Thus, from (44),

$$M_2(1) = \frac{\overline{\gamma}_{11}(1-\gamma_{21}) + \nu_2 \widetilde{\gamma}_{12}}{1-\gamma_{21}}. \tag{A.37}$$

From (A.36) it follows that $M_1(1) = (1-\gamma_{21})^{-1}$. Similarly,

$$\begin{aligned}
M_{21}(2) &= \gamma_{11}\overline{\gamma}_{11} + \overline{\gamma}_{21} \\
M_{22}(2) &= \gamma_{11}\widetilde{\gamma}_{12} \\
M_{23}(2) &= \gamma(\mathbf{c}(2), 2, 3) = \mathsf{E}(c_{1,t-1}c_{2,t-3}c_{2,t-4}c_{1,t-4}) = \gamma_{11}\gamma_{21}\widetilde{\gamma}_{12} \\
M_{24}(2) &= \gamma_{11}\mathbf{\Gamma}_2\mathbf{\Gamma}_3\frac{1}{1-\gamma_{21}}\widetilde{\gamma}_{12} = \gamma_{11}\gamma_{21}^2\widetilde{\gamma}_{12}(1-\gamma_{21})^{-1}.
\end{aligned}$$

Thus, by (44),

$$M_2(2) = \frac{(\gamma_{11}\overline{\gamma}_{11} + \overline{\gamma}_{21})(1-\gamma_{21}) + \nu_2\gamma_{11}\widetilde{\gamma}_{12}}{1-\gamma_{21}} \tag{A.38}$$

and, from (A.36),

$$M_1(2) = (1+\gamma_{11}-\gamma_{21})(1-\gamma_{21})^{-1}.$$

For any $n \geq 3$, by Lemma 5,

$$M_{21}(n) + \nu_2 M_{22}(n) = \boldsymbol{\gamma}_3'\mathbf{\Gamma}_*^{n-3}\begin{pmatrix} 0 \\ M_{21}(2) + \nu_2 M_{22}(2) \\ M_{21}(1) + \nu_2 M_{22}(1) \end{pmatrix}$$

and by (A.32),

$$\begin{aligned}
M_{23}(n) &= \boldsymbol{\gamma}_3'\mathbf{\Gamma}_*^{n-1}\boldsymbol{\gamma}(4) = \boldsymbol{\gamma}_3'\mathbf{\Gamma}_*^{n-1}\mathsf{E}\begin{pmatrix} 0 \\ 0 \\ c_{2,t-5} \end{pmatrix}(c_{1,t-5}) \\
&= \boldsymbol{\gamma}_3'\mathbf{\Gamma}_*^{n-1}\begin{pmatrix} 0, & 0, & \widetilde{\gamma}_{12} \end{pmatrix}' \\
M_{24}(n) &= \boldsymbol{\gamma}_3'\mathbf{\Gamma}_*^{n-1}\begin{pmatrix} 0, & 0, & \gamma_{21} \end{pmatrix}'\left(\frac{\widetilde{\gamma}_{12}}{1-\gamma_{21}}\right).
\end{aligned}$$

Therefore, for $n \geq 3$, applying (A.32) and (A.33) yields

$$M_2(n) = \boldsymbol{\gamma}_3'\mathbf{\Gamma}_*^{n-3}\left[\begin{pmatrix} 0 \\ M_{21}(2) + \nu_2 M_{22}(2) \\ M_{21}(1) + \nu_2 M_{22}(1) \end{pmatrix} + \nu_2\mathbf{\Gamma}_*^2\begin{pmatrix} 0 \\ 0 \\ \frac{\widetilde{\gamma}_{12}}{1-\gamma_{21}} \end{pmatrix}\right]$$

$$= \boldsymbol{\gamma}_3' \boldsymbol{\Gamma}_*^{n-3} \begin{pmatrix} 0 \\ \frac{\gamma_{11}[\overline{\gamma}_{11}(1-\gamma_{21})+\nu_2\widetilde{\gamma}_{12}]+\overline{\gamma}_{21}(1-\gamma_{21})}{1-\gamma_{21}} \\ \frac{\overline{\gamma}_{11}(1-\gamma_{21})+\nu_2\widetilde{\gamma}_{12}}{1-\gamma_{21}} \end{pmatrix}. \tag{A.39}$$

Finally, by (A.37) and (A.38),

$$M_2(n) = \boldsymbol{\gamma}_3' \boldsymbol{\Gamma}_*^{n-3} \begin{pmatrix} 0, & M_2(2), & M_2(1) \end{pmatrix}', \ n \geq 3. \dashv \tag{A.40}$$

Appendix 11. Proof of Corollary 4.

By Theorem 4**,** the autocorrelation function for the squared process has the form

$$\rho_n = \frac{\nu_2 \gamma_{S_1} \left(1 - \nu_2 \sum_{i=1}^{p} \alpha_i\right) M_2(n) - \nu_2^2 \gamma_{S_2} \left[1 - \left(1 - \nu_2 \sum_{i=1}^{p} \alpha_i\right) M_1(n)\right]}{\nu_4 \gamma_{S_1} \left(1 - \nu_2 \sum_{i=1}^{p} \alpha_i\right) - \nu_2^2 \gamma_{S_2}} \tag{A.41}$$

for $n \geq 1$. Showing that Corollary 4 is true is equivalent to demonstrating that

$$M_2(n) = (\nu_4/\nu_2) \left(1 - \left(1 - \nu_2 \sum_{i=1}^{p} \alpha_i\right) M_1(n)\right)$$

holds. Then we can show the validity of (70) by (A.41).

Without loss of generality, we only investigate the case $p = 2$. It follows from Corollary 3 that

$$M_2(i) = (\nu_4/\nu_2)\left(1 - (1 - \nu_2\alpha_1 - \nu_2\alpha_2) M_1(i)\right) \tag{A.42}$$

for $i = 1, 2$. We have to prove that (A.42) holds for any $n \geq 3$. From (59) and (60) we see that this is equivalent to showing that

$$\begin{pmatrix} \alpha_0 & \nu_2\alpha_1 & \nu_2\alpha_2 \end{pmatrix} \boldsymbol{\Gamma}_*^k \boldsymbol{\alpha}_* = 1 \tag{A.43}$$

for any k, where $\boldsymbol{\alpha}_* = \left(\alpha_0^{-1}(1 - \nu_2\alpha_1 - \nu_2\alpha_2), 1, 1\right)'$. That (A.43) holds follows from the fact that

$$(\nu_4/\nu_2) \begin{pmatrix} 0 \\ M_2(2) \\ M_2(1) \end{pmatrix} + \boldsymbol{\Gamma}_*^2 \begin{pmatrix} \alpha_0^{-1}(1 - \nu_2\alpha_1 - \nu_2\alpha_2) \\ 0 \\ (1 - \nu_2\alpha_1 - \nu_2\alpha_2)(1 - \nu_2\alpha_2)^{-1} \end{pmatrix} = \boldsymbol{\alpha}_*.$$

Note that $\mathbf{\Gamma}_*^k \boldsymbol{\alpha}_* = \boldsymbol{\alpha}_*$ for any k and $(\ \alpha_0 \quad \nu_2\alpha_1 \quad \nu_2\alpha_2\ )\boldsymbol{\alpha}_* = 1$. This completes the proof. $\dashv$