4.4.2 TGARCH model

The Threshold GARCH (TGARCH) model (Zakoïan, 1994) is defined by:

$$\sigma_t = \omega + \sum_{i=1}^{p} \left[\alpha_i \left| \varepsilon_{t-i} \right| + \gamma_i \mathit{\Pi}_{t-i}^{-} \left| \varepsilon_{t-i} \right| \right] + \sum_{j=1}^{q} \beta_j \sigma_{t-j}, \tag{4.7}$$

with $\mathit{\Pi}_t^{-}$ equal to 1 if $\varepsilon_t < 0$, and 0 otherwise.

The conditional volatility is positive when parameters satisfy $\omega > 0$, $\alpha_i \geq 0$, $\alpha_i + \gamma_i \geq 0$ and $\beta_j \geq 0$, for $i = 1, \cdots, p$ and $j = 1, \cdots, q$. Notice that ensuring covariance stationarity is not trivial, because it involves a rather non-linear constraint on parameters. In addition, this model does not nest the plain vanilla GARCH model.

4.4.3 GJR model

The *GJR* model (Glosten, Jagannathan, and Runkle, 1993) is closely related to the TGARCH model, because it is defined by:

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \left[\alpha_i \varepsilon_{t-i}^2 + \gamma_i \mathit{\Pi}_{t-i}^{-} \varepsilon_{t-i}^2 \right] + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2. \tag{4.8}$$

The conditional volatility is positive when parameters satisfy $\omega > 0$, $\alpha_i \geq 0$, $\alpha_i + \gamma_i \geq 0$ and $\beta_j \geq 0$, for $i = 1, \cdots, p$ and $j = 1, \cdots, q$. The process is covariance stationary if and only if $\sum_{i=1}^{p} (\alpha_i + \gamma_i/2) + \sum_{j=1}^{q} \beta_j < 1$ (Hentschel, 1995).

4.4.4 Cox-Box transform

Hentschel (1995) has put forward that many members of the GARCH family can be embedded in a Cox-Box transform of the form (in the case $p = q = 1$)

$$\frac{\sigma_t^{\gamma} - 1}{\gamma} = \omega + \alpha_1 \sigma_{t-1}^{\gamma} f^{\nu} (z_{t-1}) + \beta_1 \frac{\sigma_{t-1}^{\gamma} - 1}{\gamma}, \tag{4.9}$$

where $f(z_t) = |z_t - b| - c(z_t - b)$ is the news impact curve introduced by Pagan and Schwert (1990) and Engle and Ng (1993). The GARCH model is obtained from $\gamma = \nu = 2$, $b = c = 0$. The EGARCH model arises as a limit case when $\gamma = 0$, $\nu = 1$ and $b = 0$. The TGARCH model corresponds to $\gamma = \nu = 1$, $b = 0$, and $|c| \leq 1$. The GJR model is obtained from $\gamma = \nu = 2$, and $b = 0$.

For a generalized error distribution (GED) with fat-tailedness parameter ν, analyzed by Nelson (1991), we have

$$E[|z_t|] = \lambda 2^{1/\nu} \mathit{\Gamma}(2/\nu) / \mathit{\Gamma}(1/\nu),$$

where $\lambda = \left[2^{-2/\nu} \mathit{\Gamma}(1/\nu) / \mathit{\Gamma}(3/\nu) \right]^{1/2}$.